Expectation-Based Loss Aversion in Contests*

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Abstract

This paper studies a multi-player Tullock contest in which contestants exhibit reference-dependent loss aversion à la Kőszegi and Rabin (2006, 2007). Contestants may differ in their prize valuations. We verify the existence and uniqueness of pure-strategy choice-acclimating personal Nash equilibrium (CPNE) under moderate loss aversion and fully characterize the equilibrium. The equilibrium in our setting sharply contrasts that in the usual two-player symmetric case. Loss aversion can lead contestants' individual efforts to change nonmonotonically, while the total effort of the contest must strictly decrease. Further, it always leads to a more elitist distributional outcome, in the sense that a smaller set of contestants remain active in the competition and stronger contestants' equilibrium winning probabilities increase. Our results are robust under the alternative equilibrium concept of preferred personal Nash equilibrium (PPNE).

Keywords: Loss Aversion; Contest; Reference-dependent Preference; Choice-acclimating Personal Nash Equilibrium (CPNE); Preferred Personal Nash Equilibrium (PPNE).

JEL Classification Codes: C72, D72, D81, J31.

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1 Introduction

Since the seminal study of Kahneman and Tversky (1979), prospect theory has been broadly embraced as one of the most compelling alternatives to describe economic agents' risk attitude. Two behavioral notions—among others—underpin the framework: (i) economic agents derive utility from their gain and loss, which are evaluated against a reference point (*reference-dependent preferences*); and (ii) a loss reduces one's utility more than a gain of the same magnitude adds to it (*loss aversion*).[1] However, how one's reference point is determined remains elusive. Models based on conventional prospect theory assume exogenous reference points and typically fix them at status quo. Kőszegi and Rabin (2006, 2007, 2009), remarkably, propose the thesis that reference points are formed endogenously based on the agent's rational expectations about possible outcomes.[2]

Increasing evidence has been found in both the field and laboratory that provides support for the nontrivial roles played by expectations in forming reference points.[3] The notion of expectation-based loss aversion à la Kőszegi and Rabin (2006, 2007, 2009) lays a foundation for coherent and disciplined analysis of decision problems in a broad array of contexts, such as household insurance choice (Barseghyan, Molinari, O'Donoghue, and Teitelbaum, 2013); household consumption choice (Kőszegi and Rabin, 2009; Pagel, 2017); firms' marketing and pricing strategies (Herweg and Mierendorff, 2013; Heidhues and Kőszegi, 2014; Karle and Peitz, 2014; Rosato, 2016; Carbajal and Ely, 2016; Hahn, Kim, Kim, and Lee, 2018); and optimal wage schemes (Herweg, Müller, and Weinschenk, 2010). However, the numerous studies along this line have mainly focused on stand-alone decision making. The literature has paid relatively little attention to the strategic interactions between loss-averse players and understanding how their strategic choices and interplay are governed by this behavioral bias.[4]

We study a contest in which heterogeneous contestants—who are loss averse à la Kőszegi and Rabin (2006, 2007)—compete for a prize. Our interests in contests can be explained from three perspectives. First, contest-like situations are ubiquitous in social and economic

[1]Another important component of prospect theory in Kahneman and Tversky (1979) is nonlinear probability weighting. However, the economic implications of reference-dependent preferences and nonlinear probability weighting are often explored separately in the subsequent literature (Barberis, 2013; O'Donoghue and Sprenger, 2018). We abstract away nonlinear probability weighting in this paper.

[2]The study of Shalev (2000) marks an early contribution that attempts to endogenize reference points for the evaluation of gain and loss in game-theoretic settings. In contrast to Kőszegi and Rabin (2006, 2007, 2009), Shalev views the reference point as a fixed point, instead of a full distribution over all possible outcomes.

[3]Notable contributions include Abeler, Falk, Goette, and Huffman (2011), Crawford and Meng (2011), Gill and Prowse (2012), Banerji and Gupta (2014), Song (2016), Berger, Goette, and Song (2018), Goette, Graeber, Kellogg, and Sprenger (2019), among many others.

[4]The impact of loss aversion in game-theoretical settings is studied relatively more extensively in auction models. A brief review is provided later.

landscape; a plethora of competitive events exemplify a contest, ranging from electoral competitions, to military conflicts, lobbying, college admissions, and sporting events. A vast literature explores the strategic substance of contest games and the optimal contest design to attain the stated goals. The majority of these studies assumes standard preferences.[5]

Second, a contest environment provides a natural and relevant "laboratory" to examine the implications of loss aversion on players' strategic trade-off. Contestants exert nonrefundable effort to vie for limited prizes. The gambling nature of the game causes inherent uncertainty in payoffs. This naturally compels contestants—when subject to loss aversion—to deviate from the actions predicted under standard utility. The impact is more intriguing when contestants are heterogeneous: Contestants of different characteristics could respond to the behavioral bias in fundamentally different ways, as they perceive gain or loss differently due to their different expectations about the outcomes and, therefore, different reference points.

Third, contest games generate distinctively rich and intricate strategic interaction between players. As Dixit (1987) notes, players' best responses are often nonmonotone in contest games: In contrast to Cournot or Bertrand competitions, one's effort choice can be either a strategic substitute for that of another or a complement, depending on players' relative standing. It remains unclear a priori how the equilibrium deviates from that under standard preference, as this involves both the direct effect of loss aversion on individual contestants and the subsequent reflexive strategic spillover.

In this paper, we consider a multi-player Tullock contest in which contestants differ in their valuations of the prize. Kőszegi and Rabin (2007) develop the notion of *choice-acclimating personal equilibrium* (CPE) to depict the consistent behavior of expectation-based loss-averse *individuals*. Under CPE, an agent forms rational expectations about future outcomes, which shape his reference point endogenously: One's action influences future outcomes and, ultimately, his reference point; his action choice internalizes its implications for expectations and the gain or loss evaluated against the expectation-based reference point. Recently, Dato, Grunewald, Müller, and Strack (2017) and Dato, Grunewald, and Müller (2018) extend this concept to a multiple-player game theoretical model. In this paper, we follow Dato et al. (2017) and Dato et al. (2018) to focus on *choice-acclimating personal Nash equilibrium* (CPNE). We verify the existence and uniqueness of CPNE in pure strategy for moderate levels of loss aversion. This allows us to explore the ramifications of loss aversion for contestants' incentives.[6]

[5]Notable exceptions include Müller and Schotter (2010); Gill and Prowse (2012); and Dato, Grunewald, and Müller (2018), who also assume expectation-based loss-averse players.

[6]Gill and Stone (2010) and Dato, Grunewald, and Müller (2018) show that CPNE may cease to exist when contestants are highly loss averse. We focus on the case of moderate loss aversion; the analysis of the contest game under strong loss aversion is provided in the online appendix.

Gill and Prowse (2012) and Dato, Grunewald, and Müller (2018) show that the CPNE in a two-player symmetric contest coincides with the Nash equilibrium (NE) under standard preferences. In contrast, we demonstrate that loss aversion significantly varies contestants' incentives and the equilibrium interplay when they are heterogeneous and/or the number of contestants exceeds two. Our observations can be summarized as follows.

i. In a two-player asymmetric contest, when contestants are heterogeneous, loss aversion leads the weaker contestant to unambiguously decrease his effort, while the stronger may either increase or decrease his effort, depending on the distribution of prize valuations. We show that the stronger decreases his effort when competition is more lopsided, i.e., when contestants' prize valuations are sufficiently dispersed.

ii. When the contest involves three or more symmetric contestants, they uniformly reduce their efforts when loss aversion is present.

iii. Loss aversion triggers heterogeneous responses from asymmetric contestants when more than two asymmetric contestants are involved. Bottom contestants are discouraged: They reduce their efforts and may even drop out of the competition by placing a zero bid. Subtler effects, however, loom large for those in the upper bracket: They may either increase or decrease their efforts, and their responses can be nonmonotone, in the sense that the top contestant slackens off, while those in the middle step up their bids. Despite the complexity, our analysis obtains a complete account of the incentive effects.

iv. Despite the mixed responses in individual equilibrium efforts, we obtain unambiguous observations about the effect of loss aversion on aggregate incentive and distribution. The analysis predicts that overall effort always drops, regardless of the diverging responses of individual contestants. A more *elitist* redistribution pattern may arise: Loss aversion leads to a smaller set of active contenders, and stronger contestants always end up with higher winning odds.

We now provide a brief account of the incentive effect and strategic implications of loss aversion. Loss aversion generates disutility to contestants, which causes their behavior to deviate from the Nash equilibrium under standard preferences: They must internalize the disutility in their effort choice, and the equilibrium must strike a balance between the material utility derived from the contest and the psychological gain-loss (dis)utility. Expectation-based loss aversion in the contest causes a disutility proportional to $p_i(1-p_i)$, where p_i is contestant i's probability of winning, and therefore the term literally measures the uncertainty expected by the contestant. Let us begin with a simple case of two heterogeneous contestants. Two effects would arise in the contest game.

First, there is a *(direct) uncertainty-reducing effect* caused by contestants' loss aversion. As the disutility is proportional to $p_i(1-p_i)$—i.e., the measure of uncertainty—contestants are compelled to reduce the uncertainty in terms of outcomes. The weaker contestant tends to decrease his effort: He expects a relatively pessimistic outcome—i.e., smaller winning odds—which leads him to cut back on his effort to protect himself from ex post unrewarded input. Clearly, this further reduces his winning odds and reduces uncertainty, because $p_i(1-p_i)$ decreases with p_i for $p_i < \frac{1}{2}$. In contrast, the stronger contestant would expect a more optimistic outcome; under loss aversion, he tends to increase effort to prevent losing unexpectedly to his weaker opponent: A higher effort increases p_i and decreases uncertainty.

Second, there is an *(indirect) competition effect* caused by the strategic interactions between contestants in the game: When the uncertainty-reducing effect causes each individual contestant to adjust his effort choice, his opponents must respond strategically. Recall the aforementioned nonmonotone best response correspondence in contest: A contestant's effort is a strategic complement to that of his opponent when he is in the lead, while it is a strategic substitute when he is behind (see Dixit, 1987). When the uncertainty-reducing effect encourages the favorite to step up his effort, the underdog is further discouraged, as he expects smaller odds to win: Both the (indirect) competition effect and the (direct) uncertainty-reducing effect lead the underdog to concede further. In contrast, when the underdog cut back on his effort, the favorite is tempted to reduce his effort in response, as a less competitive opponent allows the latter to slack off without suffering lower winning odds: The two effects oppose each other, and the overall effect of loss aversion is ambiguous.

We show that the stronger player increases his effort when the asymmetry in the contest remains mild, in which case the uncertainty-reducing effect prevails. When the contest is excessively asymmetric, however, the competition effect overshadows the uncertainty-reducing effect, which decreases his effort. In the knife-edge case of two-player symmetric contests, both effects vanish, and CPNE coincides with NE (see, also, Gill and Stone, 2010; Gill and Prowse, 2012; and Dato, Grunewald, and Müller, 2018). Symmetry leads each to win with a probability 1/2: The marginal effect of a variation in p_i on $p_i(1-p_i)$ degenerates to zero, which nullifies the uncertainty-reducing effect and, in turn, defuses the indirect competition effect.

This rationale extends to the case of $N \geq 3$ contestants. Consider a multi-player contest with homogeneous contestants. Despite the symmetry between individual contestants, all of them are underdogs in the competition, as each must outperform a collection of equally competitive opponents and stands a chance of only $1/N$ to win the prize. The uncertainty-reducing effect thus compels each of them to reduce effort. The competition effect in fact catalyzes a conflicting force because one, as an underdog, would be encouraged to step up efforts when others concede. The second-order competition effect, however, is insufficient for a reversal. When contestants are asymmetric, the asymmetry substantially complicates the analysis and yields subtler incentive and strategic implications. However, a rationale based on the tension between the two fundamental effects continues to provide a lucid and intuitive account of the observations. We elaborate on this in Section 3.2.

Following the notion of preferred personal equilibrium (Kőszegi and Rabin, 2006, 2007) for individual decision making, Dato, Grunewald, Müller, and Strack (2017) and Dato, Grunewald, and Müller (2018) propose an alternative equilibrium concept, i.e., the preferred personal Nash equilibrium (PPNE). A CPNE requires that one's reference point fully adapt to actual action choice (i.e., choice acclimating), while a PPNE assumes fixed expectations (i.e., choice unacclimating) and requires that agents follow their most preferred credible action plan. These equilibrium concepts, CPNE and PPNE, arguably apply to different contexts. However, we show that our main results obtained under CPNE are qualitatively robust when an alternative equilibrium concept is adopted.

Related Literature Our paper contributes to the growing literature on the strategic interaction between loss-averse economic agents in the sense of Kőszegi and Rabin (2006, 2007, 2009). Gill and Stone (2010) and Dato, Grunewald, and Müller (2018) pioneer the study of contests/tournaments with the presence of expectation-based loss aversion.[7] Both studies consider two-player simultaneous-move rank-order tournament models and primarily focus on the fundamentals of equilibria in the games.[8] Our study differs from these in both setting and focus. We consider a multi-player Tullock contest with heterogeneous contestants and provide a comprehensive account of the impact of loss aversion on contestants' incentives and equilibrium outcomes.

A handful of studies incorporate expectation-based loss aversion into auction models. Lange and Ratan (2010) show that predictions on bidders' behavior largely depend on whether the auctioned items and money are consumed along the same dimension. Eisenhuth and Grunewald (2018) compare first-price auctions to all-pay auctions, and show that the revenue ranking also depends sensitively on how individuals evaluate gain and loss. Rosato and Tymula (2019) provide experimental evidence for the difference in bidding behavior in real-item auctions vis-à-vis induced-value auctions. Balzer and Rosato (2020) study common-value auctions, while Rosato (2019) analyzes sequential auctions. Mermer (2017) investigates optimal revenue-maximizing prize allocation in an all-pay auction model, and shows that a contest designer may prefer to split her prize purse into several uniform prizes when contestants are loss averse. Eisenhuth (2019) studies a revenue-maximizing mechanism; he shows

[7]Relatedly, Gill and Stone (2015) and Daido and Murooka (2016) adopt reference-dependent preferences in models of team production.

[8]Gill and Prowse (2012) present a theoretical model in which contestants move sequentially. Rosato (2017) considers a sequential negotiation model that allows for a loss-averse buyer.

that the optimal auction is an all-pay auction with a minimum bid when gain and loss are evaluated in separable dimensions. Studies in auction models typically assume incomplete information and ex ante symmetric bidders, which yield equilibrium bidding strategy as functions of bidders' private types. In contrast, we consider a complete-information Tullock contest model. A pure-strategy equilibrium exists in which each contestant bids a fixed amount of effort. This setting allows us to model ex ante asymmetric competition and explore explicitly the implications of loss aversion for strategic interactions between heterogeneous players.

Our paper also contributes to the thin literature on contests with behavioral abnormality. Anderson, Goeree, and Holt (1998) allow for boundedly rational bidders in all-pay auctions. Baharad and Nitzan (2008) provide a rationale for rent under-dissipation based on probability distortion. Cornes and Hartley (2012); Müller and Schotter (2010); and Chen, Ong, and Segev (2017) introduce non-expectation-based loss aversion in contest models.

The remainder of the paper is organized as follows. Section 2 sets up the model and presents a preliminary analysis that establishes the existence and uniqueness of CPNE in a generalized lottery contest model under moderate loss aversion. Section 3 characterizes the equilibrium under a more specific contest technology and examines the impact of expectation-based loss aversion on contestants' incentives and equilibrium outcomes. Section 4 conducts an analysis of the contest game based on the alternative equilibrium concept of PPNE and discusses the robustness of predictions obtained under CPNE, and Section 5 concludes. All proofs are relegated to the appendix; an online appendix presents an analysis of the contest game under strong loss aversion.

2 Model and Preliminaries

There are $N \geq 2$ contestants competing for a prize. The prize bears a value v_i for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, N\}$, which is common knowledge. Without loss of generality, we assume $v_1 \geq \ldots \geq v_N > 0$.

2.1 Winner-selection Mechanism

Contestants simultaneously exert irreversible and nonnegative efforts to compete for the prize. We consider a generalized lottery contest, with its winner being selected through a ratio-form contest success function: For a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_N)$, a contestant

i wins with a probability

$$p_i(\boldsymbol{x}) = \begin{cases} \frac{f_i(x_i)}{\sum_{j=1}^N f_j(x_j)} & \text{if } \sum_{j=1}^N x_j > 0, \\ \frac{1}{N} & \text{if } \sum_{j=1}^N x_j = 0, \end{cases} \tag{1}$$

where the function $f_i(\cdot)$ converts one's effort entry into his effective bid in the lottery and is typically labeled the *impact function* in the contest literature. We impose the following conditions on the set of impact functions $\{f_i(\cdot)\}_{i=1}^N$.

Assumption 1 *$f_i(\cdot)$ is a twice-differentiable function, with $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$, and $f_i(0) = 0$.*

Jia (2008) and Fu and Lu (2012) demonstrate that the generalized lottery contest model is underpinned by a unique noisy ranking system. Imagine that contestants are evaluated through a set of noisy signals of their performance $\boldsymbol{z} := (z_1, \ldots, z_N)$. Following the discrete choice framework of McFadden (1973, 1974),[9] the noisy signal z_i is assumed to be described by

$$\ln z_i = \ln f_i(x_i) + \varepsilon_i, \, \forall\, i \in \mathcal{N},$$

where the function $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the deterministic component of contestant i's output[10] and the additive noise term ε_i reflects the randomness in the production or evaluation process. Idiosyncratic noises $\boldsymbol{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_N)$ are independently and identically distributed, being drawn from a type I extreme-value (maximum) distribution, with a cumulative distribution function

$$\Gamma(\varepsilon_i) = \exp\left[-\exp\left(-\varepsilon_i\right)\right], \varepsilon_i \in (-\infty, +\infty), \, \forall\, i \in \mathcal{N}.$$

A complete ranking of contestants immediately results when the shocks $\boldsymbol{\varepsilon}$ are realized. A contestant i secures the prize if and only if he obtains the top rank, i.e., $z_i > \max_{j \neq i}\{z_j\}$, which occurs with a probability specified by Equation (1).

2.2 Contestants' Preferences

Contestants are assumed to be expectation-based loss averse, as in Kőszegi and Rabin (2006). To put this formally, fixing opponents' effort profile $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, contestant i's expected payoff of exerting x_i when he expects himself to exert effort $\hat{x}_i$, de-

[9]The framework of McFadden's discrete choice model is further introduced and studied in various respects by works collected in Manski and McFadden (1981).

[10]Define $\ln f_i(x_i) = -\infty$ if $f_i(x_i) = 0$.

noted by $U_i(x_i, \hat{x}_i, \boldsymbol{x}_{-i})$, is given by

$$\begin{aligned} U_i(x_i, \hat{x}_i, \boldsymbol{x}_{-i}) =& p_i(x_i, \boldsymbol{x}_{-i}) \times \Big\{ v_i + \eta \left[1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})\right] \times \mu(v_i) \Big\} \\ &+ \left[1 - p_i(x_i, \boldsymbol{x}_{-i})\right] \times \big\{0 + \eta p_i(\hat{x}_i, \boldsymbol{x}_{-i}) \times \mu(-v_i)\big\} - x_i + \eta\mu(\hat{x}_i - x_i), \quad (2) \end{aligned}$$

where the parameter $\eta \geq 0$ is the weight a contestant attaches to his gain-loss utility relative to his material utility; $\mu(\cdot)$ is the universal psychological gain-loss utility and is defined as the following:

$$\mu(c) = \begin{cases} c & \text{if} \quad c \geq 0, \\ \lambda c & \text{if} \quad c < 0. \end{cases}$$

The parameter λ is assumed to exceed one, which captures a contestant's loss aversion in the sense that he is more sensitive to a loss than to a gain of the same magnitude.

By Equation (2), the contestant, when expecting himself to exert an effort $\hat{x}_i$,[11] would perceive a gain of $\mu(\hat{x}_i - x_i) = \hat{x}_i - x_i > 0$ when his effort x_i is below his expectation $\hat{x}_i$, and sense a loss of $\big|\mu(\hat{x}_i - x_i)\big| = \lambda|\hat{x}_i - x_i|$ otherwise. Furthermore, he expects himself to win with probability $p_i(\hat{x}_i, \boldsymbol{x}_{-i})$ and lose with probability $1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})$. This forms his stochastic reference point along the prize dimension. A contestant compares the realized outcome of the contest with each possible outcome in the reference lottery. In particular, winning the contest feels like a gain of $[1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})] \times \mu(v_i)$, while losing it generates a loss of $p_i(\hat{x}_i, \boldsymbol{x}_{-i}) \times \big|\mu(-v_i)\big|$.

Note that we assume that contestants evaluate prize and effort separately when deriving expression (2). Lange and Ratan (2010) consider first-price and second-price auctions with expectations-based loss-averse bidders. They contend that model predictions differ substantially when bidders evaluate their gain and loss from money and the auction item separately vis-à-vis when they evaluate them jointly based on the net utility of the transaction. In contrast, our results would be immune to such modeling nuances due to the all-pay feature of the contest game whenever contestants play pure strategies.[12]

[11] We restrict our attention to pure strategy without loss of generality. It can be shown that a choice-acclimating Nash equilibrium in mixed strategies does not exist by an argument similar to the proof of Proposition 3 in Dato, Grunewald, and Müller (2018).

[12] In a first-price or second-price auction, a bid incurs a cost if and only if one wins. The evaluation of gain and loss in different outcomes thus depends on whether the auction item and money are consumed in separable dimensions of consumption space. In contrast, a contest requires a nonrecoverable bid, and the effort cost is sunk irrespective of the realized outcome. Therefore, the evaluation is independent of the nuance.

2.3 Equilibrium Concepts

As stated in the Introduction, our analysis primarily focuses on the solution concept of CPNE. The notion of CPE (Kőszegi and Rabin, 2007) requires that reference points be formed through rational expectations, and a loss-averse agent's action choice fully internalizes its impact on his expectations and the gain-loss utility measured against the expectation-based reference point. Dato, Grunewald, Müller, and Strack (2017) and Dato, Grunewald, and Müller (2018) integrate the notion into analysis of strategic interaction between loss-averse players and develop the solution concept of CPNE. In our context, a CPNE is formally defined as follows.

Definition 1 ***(Choice-acclimating personal Nash equilibrium)*** *The effort profile* $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_N^*)$ *constitutes a choice-acclimating personal Nash equilibrium (CPNE) in pure strategy if for all* $i \in \mathcal{N}$*,*

$$U_i(x_i^*, x_i^*, \boldsymbol{x}_{-i}^*) \geq U_i(x_i, x_i, \boldsymbol{x}_{-i}^*), \text{ for all } x_i \in [0, \infty).$$

By Definition 1, each contestant's expectations about future outcomes will have fully adapted to his actual strategic choice when the uncertainty is resolved; he then commits to a strategy that maximizes his expected utility given his opponents' strategy profile. In other words, the expectation is choice acclimating. The notion of choice acclimating, according to Kőszegi and Rabin (2007), is more plausible when the action is chosen long before the outcome of the contest is realized, and thus each contestant's belief can eventually be adapted to the effort level he has chosen.[13]

2.4 Equilibrium Existence and Uniqueness

A CPNE requires $x_i = \hat{x}_i$ for all $i \in \mathcal{N}$. Integrating the condition into expression (2) and carrying out the algebra yield

$$\widehat{U}_i(x_i, \boldsymbol{x}_{-i}) := U_i(x_i, x_i, \boldsymbol{x}_{-i}) = \underbrace{p_i(x_i, \boldsymbol{x}_{-i})v_i - x_i}_{\text{material utility}} \underbrace{- \eta(\lambda - 1)p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i}_{\text{gain-loss utility}}. \quad (3)$$

From the above expression, it is obvious that our setup degenerates to a standard contest model if the second term vanishes, i.e., if $\eta(\lambda - 1) = 0$. For notational convenience, let us denote $\eta(\lambda - 1)$ by k. This is the overall weight in the contestant's expected utility attached

[13] Kőszegi and Rabin (2006, 2007) propose another equilibrium concept, the *(preferred) personal equilibrium*, to depict the scenario in which a player makes his decision shortly before the outcome is realized, which prevents his past expectations from being adapted to his actual action choice, i.e., contestants' expectations are choice-unacclimating. Our main results are robust to this alternative equilibrium concept. See Section 4 for more discussion.

to the net loss caused by loss aversion (see also Herweg, Müller, and Weinschenk, 2010; Dato, Grunewald, and Müller, 2018), and hence can be viewed as a composite measure of the intensity of contestants' reference-dependent loss aversion.

Simple math verifies that a contestant's expected utility $\widehat{U}_i(\cdot)$ is strictly concave in his effort x_i for $k \leq \frac{1}{2}$.[14] A contestant's effort choice can therefore be pinned down by the prevailing first-order condition. Denote by $BR_i(\boldsymbol{x}_{-i})$ a contestant i's best response, which can be derived as the following:

$$BR_i(\boldsymbol{x}_{-i}) = \begin{cases} 0 & \text{if } \left.\frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i}\right|_{x_i=0} \leq 0, \\ \text{the unique solution to } \frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = 0 & \text{otherwise.} \end{cases}$$

A CPNE is thus an effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_N)$ with $x_i = BR_i(\boldsymbol{x}_{-i})$ for all $i \in \mathcal{N}$.

Szidarovszky and Okuguchi (1997); Stein (2002); and Cornes and Hartley (2005) establish the existence and uniqueness of Nash equilibria in the contest game under standard preferences, which corresponds to the case of $k = 0$ in our setup. We now demonstrate that this result can be retained when contestants are moderately loss averse à la Kőszegi and Rabin (2007).

Theorem 1 ***(Existence and uniqueness of CPNE with moderate loss aversion)*** *Suppose that Assumption 1 is satisfied and $k \equiv \eta(\lambda - 1) \in [0, \frac{1}{3}]$. Then there exists a unique pure-strategy CPNE of the contest game.*

Theorem 1 requires moderate loss aversion. It is well known in the literature that a CPNE may fail to exist when contestants are excessively loss averse.[15] Our analysis mainly focuses on the case of $k \leq 1/3$; the implications of a large k will be discussed in an online appendix.

3 Equilibrium Analysis

In this section, we characterize the unique CPNE in the contest game and delineate how expectation-based loss-averse contestants' incentive and behavior depart from those of their counterparts with standard preferences. To gain more mileage, we focus on the popularly adopted lottery contest model with linear impact function (see Stein, 2002; Franke, Kanzow,

[14] To see this, note that $\frac{\partial^2 \widehat{U}_i}{\partial x_i^2} = (1-p_i)p_i v_i \times \left\{(1 - k + 2kp_i)\frac{f_i''(x_i)}{f_i(x_i)} + (-2 + 4k - 6kp_i)p_i\left[\frac{f_i'(x_i)}{f_i(x_i)}\right]^2\right\}$. Moreover, we have $f_i''(x_i) \leq 0$ from Assumption 1; and $-2+4k-6kp_i \leq -6kp_i < 0$ for $k \leq \frac{1}{2}$ and $p_i > 0$. Therefore, $\frac{\partial^2 \widehat{U}_i}{\partial x_i^2} < 0$ for $x_i > 0$ if $k \leq \frac{1}{2}$.

[15] See Dato, Grunewald, Müller, and Strack (2017) and Dato, Grunewald, and Müller (2018) for detailed discussion of the nonexistence of CPNE.

Leininger, and Schwartz, 2013, among many others). The following assumption is imposed throughout the rest of the section.

Assumption 2 *$f_i(x_i) = x_i$ for all $i \in \mathcal{N}$.*

Denote by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_N^*)$ the equilibrium effort profile in the unique CPNE, which is fully characterized in the next result.

Proposition 1 ***(Characterization of equilibrium effort profile)*** *Suppose that Assumption 2 is satisfied and $k \equiv \eta(\lambda - 1) \in \left[0, \frac{1}{3}\right]$. In the unique CPNE, contestant i's equilibrium effort entry x_i^*, with $i \in \mathcal{N}$, is given by*

$$x_i^* = g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \leq s, \\ \frac{\sqrt{(1-3k)^2 s^2 + 8ks^2\left(1-k-\frac{s}{v_i}\right)} - (1-3k)s}{4k} & \text{otherwise,} \end{cases} \tag{4}$$

where $s > 0$ is the unique solution to $\sum_{i=1}^N g_i(s) = s$.

Next, we investigate the impact of reference-dependent preferences on contestants' equilibrium effort choice. For this purpose, we slightly abuse the notation and write x_i^* as $x_i^*(k)$—i.e., a function of k—to highlight the relationship between equilibrium effort and the degree of loss aversion.

3.1 Contests with Two Contestants: $N = 2$

Although the equilibrium effort profile $\boldsymbol{x}^*(k) := \big(x_1^*(k), \ldots, x_N^*(k)\big)$ is fully characterized in Proposition 1, a closed-form solution is unavailable in general because s is implicitly determined by the condition $\sum_{i=1}^N g_i(s) = s$. To provide a lucid account of the impact of reference-dependent preferences on equilibrium outcomes, it is useful to first restrict our attention to a two-player case, as in the literature (e.g., Gill and Stone, 2010, 2015; Dato, Grunewald, Müller, and Strack, 2017).

Assuming $N = 2$, the equilibrium effort profile $\big(x_1^*(k), x_2^*(k)\big)$ can be solved explicitly as the following:

Proposition 2 *Suppose that Assumption 2 is satisfied, $k \in [0, \frac{1}{3}]$, and $N = 2$. The equilibrium effort pair $\big(x_1^*(k), x_2^*(k)\big)$ is given by*

$$x_1^*(k) = \frac{\theta}{(1+\theta)^2} v_1 - \frac{\theta(1-\theta)}{(1+\theta)^3} k v_1,$$

and

$$x_2^*(k) = \frac{1}{(1+\theta)^2}v_1 - \frac{1-\theta}{(1+\theta)^3}kv_1,$$

where

$$\theta = \frac{1}{2}\left[\left(\frac{v_1}{v_2}-1\right)\times\frac{1+k}{1-k}+\sqrt{\left(\frac{v_1}{v_2}-1\right)^2\times\left(\frac{1+k}{1-k}\right)^2+\frac{4v_1}{v_2}}\right].$$

A closer look at the equilibrium result leads to the following comparative statics.

Proposition 3 ***(Impact of reference-dependent preferences on incentives in two-player contests)*** *Suppose that Assumption 2 is satisfied and $N = 2$. The following statements hold:*

i. If $v_1 = v_2 =: v$, then $x_1^*(k) = x_2^*(k) = \frac{1}{4}v$ and hence $\frac{dx_1^*}{dk}\big|_{k=0} = \frac{dx_2^*}{dk}\big|_{k=0} = 0$.

ii. If $v_1 > v_2$, then $\frac{dx_2^*}{dk}\big|_{k=0} < 0$. Moreover, $\frac{dx_1^*}{dk}\big|_{k=0} > 0$ if and only if $\frac{v_1}{v_2} < 3$.

Part (i) of Proposition 3 states that when contestants are homogeneous, the unique pure-strategy CPNE is symmetric and identical to the unique Nash equilibrium for contestants with standard preferences. This observation echoes the findings of Gill and Stone (2010, Proposition 2) and Dato, Grunewald, Müller, and Strack (2017, Proposition 1) in alternative contest settings. However, part (ii) of Proposition 3 demonstrates that loss aversion plays a significant role for heterogeneous contestants, which causes the predictions to diverge from those in a standard framework. Loss aversion reduces the weak contestant's equilibrium bid; in contrast, the strong contestant may either increase or decrease his effort, depending on the degree of heterogeneity between the contestants. When the dispersion of contestants' prize valuations remains moderate, i.e., $v_1/v_2 < 3$, the strong contestant exerts more effort under loss aversion; he nevertheless decreases his effort level when the competition is excessively asymmetric, i.e., $v_1/v_2 > 3$.

Intuition and Decomposition: Two Effects To elaborate on the change in incentive triggered by expectation-based loss aversion, it is useful to reexamine a contestant's utility function. Recall that when contestants' expectations are choice-acclimating, one's utility is given by

$$\widehat{U}_i(x_i, \boldsymbol{x}_{-i}) = \underbrace{p_i(x_i, \boldsymbol{x}_{-i})v_i - x_i}_{\text{material utility}}\underbrace{-kp_i(x_i, \boldsymbol{x}_{-i})[1-p_i(x_i, \boldsymbol{x}_{-i})]v_i}_{\text{gain-loss utility}}.$$

The psychological gain-loss utility is proportional to $p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]$, which can be viewed as a natural measure of the uncertainty regarding the outcome of the contest. A

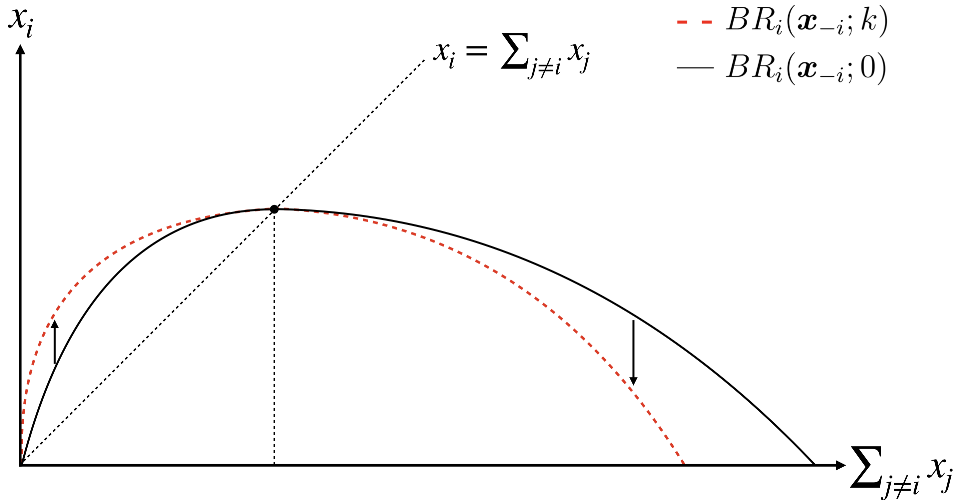


Figure 1: Best Response Function for Contestant i: $x_i = BR_i(\boldsymbol{x}_{-i}; k)$.

loss-averse contestant—i.e., $k > 0$—naturally dislikes uncertainty, which compels him to take proactive action to reduce it. We are now ready to decompose the incentive effect into two sources.

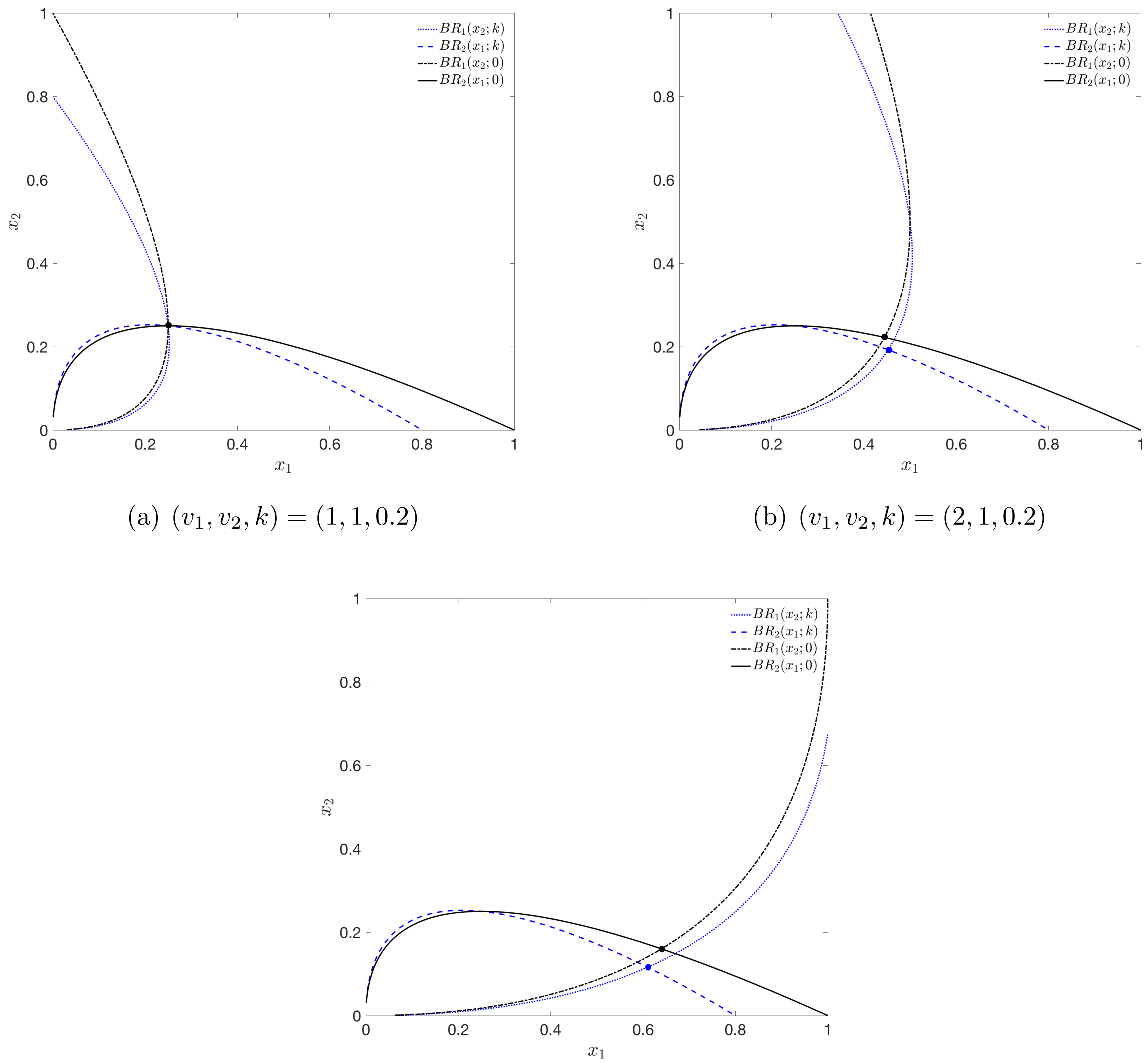
First, a *(direct) uncertainty-reducing effect* caused by expectation-based loss aversion is immediate. Note that the uncertainty measure, $p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]$, strictly increases with $p_i(x_i, \boldsymbol{x}_{-i})$ first, reaches its maximum when $p_i(x_i, \boldsymbol{x}_{-i}) = 1/2$, and then strictly decreases. A loss-averse contestant, to reduce the uncertainty about the outcome, is tempted to decrease (increase) his effort if his winning probability falls below (exceeds) the threshold $1/2$: The underdog—with $p_i(x_i, \boldsymbol{x}_{-i}) < 1/2$—is poised to drive down $p_i(x_i, \boldsymbol{x}_{-i})$ toward zero, while the favorite—with $p_i(x_i, \boldsymbol{x}_{-i}) > 1/2$—would push it toward one. To put this more intuitively, the underdog expects a less likely win—which compels him to reduce unnecessary efforts—while the favorite steps up his effort to insure against an inadvertent loss.

To put this more formally, define $\widetilde{U}_i(x_i, \boldsymbol{x}_{-i}) := -\, kp_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i$, which is a contestant i's gain-loss utility. It follows immediately that

$$\frac{\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = -k\left[1 - 2p_i(x_i, \boldsymbol{x}_{-i})\right]\frac{\partial p_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} v_i.$$

The term $\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})/\partial x_i$ measures one's marginal benefit when taking proactive action—i.e., adjusting his effort choice—to improve his gain-loss utility, and also the strength of the uncertainty-reducing effect. The sign of $\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})/\partial x_i$ depends solely on the difference between $p_i(x_i, \boldsymbol{x}_{-i})$ and $\frac{1}{2}$, or equivalently in our context, the comparison between the effort of contestant i—i.e., x_i—and the aggregate effort of all his opponents, i.e., $\sum_{j\neq i} x_j$. If $x_i < \sum_{j\neq i} x_j$, then $p_i(x_i, \boldsymbol{x}_{-i}) < \frac{1}{2}$, and the second term turns negative, which implies that loss aversion tends to disincentivize a contestant; conversely, it would further incentivize the contestant if $x_i > \sum_{j \neq i} x_j$ and $p_i(x_i, \boldsymbol{x}_{-i}) > \frac{1}{2}$.

The effect is illustrated with Figure 1, which plots a contestant's best response in the contest game with and without loss aversion. The presence of loss aversion causes an inward rotation of the best response curve: With $k > 0$, a contestant steps up his bid in his best response to a given $\sum_{j \neq i} x_j$ for $x_i > \sum_{j \neq i} x_j$; he backs off for $x_i < \sum_{j \neq i} x_j$.



(a) $(v_1, v_2, k) = (1, 1, 0.2)$

(b) $(v_1, v_2, k) = (2, 1, 0.2)$

(c) $(v_1, v_2, k) = (4, 1, 0.2)$

Figure 2: Equilibrium Effort Profiles: $\big(x_1^*(k), x_2^*(k)\big)$ and $\big(x_1^*(0), x_2^*(0)\big)$.

The uncertainty-reducing effect further triggers an *(indirect) competition* effect that comes into play through the reflexive interaction between contestants, as each contestant must adjust his effort choice in response to the change in the effort of his opponent. Such indirect effect mainly stems from contestants' trade-offs in terms of the material utility in the contest. To be more specific, a contestant must rebalance his material gain of $p_i(x_i, \boldsymbol{x}_{-i})v_i$ and cost x_i in response to a change in $\boldsymbol{x}_{-i}$, which is no different from the strategic interaction in a standard contest. Dixit (1987) elaborates on the nonmonotone best-response functions caused by the particular (material) payoff structure in contests, which is also depicted in Figure 1: Opponents' efforts are strategic complements to a contestant i if he is in the lead (i.e., $x_i > \sum_{j \neq i} x_j$) and being strategic substitutes otherwise. In our context, on the one hand, a more aggressive favorite—due to the uncertainty-reducing effect—further disincentivizes the underdog because of the strategic substitutability of efforts, as a win is even less likely for the underdog; on the other hand, the concession of the underdog allows the favorite to slack off because of the strategic complementarity, as a lower effort may still render him an equally likely win. The former complements the uncertainty-reducing effect for the underdog, while the latter conflicts with the uncertainty-reducing effect (see Table 1) for the favorite.

This rationale sheds immediate light on the knife-edge case of symmetric two-player contests, in which each contestant wins with an equal probability in the unique equilibrium. The marginal impact of effort on the gain-loss utility—i.e., $\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})/\partial x_i$—boils down to zero. Therefore, the (direct) uncertainty-reducing effect vanishes on the margin, which also defuses the (indirect) competition effect. Contestants thus behave as if under standard preferences, which leads to the prediction of part (i) of Proposition 3, as in Gill and Stone (2010, Proposition 2) and Dato, Grunewald, Müller, and Strack (2017, Proposition 1).

When contestants are heterogeneous, the uncertainty-reducing effect arises, which in turn triggers the competition effects. The CPNE thus deviates from the Nash equilibrium under standard preferences. Table 1 provides a summary of these effects for each contestant. It is evident that both effects tend to weaken the underdog's effort incentive, and hence the underdog would reduce his effort unambiguously. In contrast, Proposition 2 demonstrates that loss aversion may lead the strong contestant to reduce his effort when the dispersion between contestants' prize valuations is sufficiently large, i.e., $v_1/v_2 > 3$—in which case the competition effect outweighs the uncertainty-reducing effect.

	Uncertainty-reducing effect	Competition effect	Aggregate effect
Weak player	$\downarrow$	$\downarrow$	$\downarrow$
Strong player	$\uparrow$	$\downarrow$	$\downarrow$ or $\uparrow$

Table 1: Decomposition of the Impact of Reference-dependent Preferences on Incentive.

Role of Asymmetry and Incentive of the Favorite We now elaborate on how the tension between these competing forces for the strong player subtly depends on the degree of asymmetry in the competition. Denote by x_i and x_j, respectively, the effort entry of the indicative contestant and his opponent. We first demonstrate that $\partial\widetilde{U}_i(x_i,x_j)/\partial x_i$ is nonmonotone in x_i: The uncertainty-reducing effect is poised to diminish for the strong contestant when the competition becomes increasingly lopsided, while it tends to strengthen for the weak.

Carrying out the algebra, we have that

$$\begin{aligned}\frac{\partial^2\widetilde{U}_i(x_i,x_j)}{\partial x_i^2} &= -k\left\{-2\left[\frac{\partial p_i(x_i,x_j)}{\partial x_i}\right]^2+\left[1-2p_i(x_i,x_j)\right]\frac{\partial^2 p_i(x_i,x_j)}{\partial^2 x_i}\right\}v_i\\ &= \frac{2kx_j}{(x_i+x_j)^4}\left(2x_j-x_i\right)v_i.\end{aligned}$$

That is, when an opponent's effort x_j is relatively large, i.e., $2x_j-x_i>0$, a contestant's gain-loss utility $\widetilde{U}(x_i,x_j)$ is convex in x_i, with $\partial^2\widetilde{U}_i(x_i,x_j)/\partial x_i^2>0$; when x_j is small compared with x_i, i.e., $2x_j-x_i<0$, $\widetilde{U}(x_i,x_j)$ turns concave in x_i, with $\partial^2\widetilde{U}_i(x_i,x_j)/\partial x_i^2<0$. A large (small) effort gap is arguably the outcome of a less (more) even contest, i.e., when v_1 is excessively (moderately) large relative to v_2. When contestant i is the underdog, i.e., $x_i<x_j$, convexity implies that $\left|\partial\widetilde{U}_i(x_i,x_j)/\partial x_i\right|$ enlarges when he further decreases his effort so as to reduce uncertainty; the direct effect for the underdog strengthens itself. This effect is particularly significant in the extreme case of x_i close to zero, in which case the contestant is exceptionally weak. However, the same does not hold for the strong contestant. Although $\left|\partial\widetilde{U}_i(x_i,x_j)/\partial x_i\right|$ increases with x_i when x_i is above x_j but remains below $2x_j$, it starts to diminish once x_i exceeds the threshold. This implies that the favorite perceives a declining marginal benefit from his uncertainty-reducing effort—i.e., a diminishing uncertainty-reducing effect—when he possesses excessive advantage.

Next, we demonstrate that increasing asymmetry magnifies the competition effect. Figure 2 depicts contestants' best responses in three scenarios, with and without loss aversion. It is straightforward to observe that the best-response correspondence is concave, which implies more sensitive strategic responses—or, in other words, stronger strategic dependence—when the relative difference in contestants' efforts—i.e., x_1/x_2—is large; conversely, it vanishes when efforts are sufficiently close. It is thus intuitive to conclude that the competition effect strengthens when contestants' prize valuations differ more significantly.

Our result can thus be interpreted in light of these observations. When the contest is increasingly asymmetric—i.e., when v_1 increases substantially relative to v_2—the direct uncertainty-reducing effect for the strong contender (i.e., contestant 1) diminishes by itself, which prevents the contestant from sharply increasing his effort; in contrast, the indirect competition effect strengthens, which compels him to decrease his effort more. The competition effect is thus poised to outweigh the uncertainty-reducing effect for the favorite when the contest is more imbalanced. Proposition 3(ii) states that the derivative $(dx_1^*/dk)\big|_{k=0}$ turns negative when $v_1/v_2 > 3$.

Before we conclude this subsection, it is noteworthy that the strong contestant's mixed response stands in contrast to the observation obtained in settings of incomplete-information all-pay auction with ex ante symmetric players. Müller and Schotter (2010) and Mermer (2017) both witness a bifurcation effect, in that the (interim) stronger contestants step up their efforts and their (interim) weaker counterparts do the opposite.

3.2 Contests with Three or More Contestants: $N \geq 3$

We now extend the analysis to contests with three or more contenders. Additional players significantly enrich the game and yield substantially more complex strategic interactions. Our analysis begins with a simple case of symmetric players with $v_i = v > 0$, $\forall\, i \in \mathcal{N}$. We demonstrate that in contrast to the symmetric two-player contest, the CPNE departs from the Nash equilibrium under standard preferences, despite the symmetry between contestants.

Proposition 4 ***(Equilibrium in contest with three or more homogeneous players)*** *Suppose that the contest involves $N \geq 3$ homogeneous contestants with $v_1 = \ldots = v_N =: v > 0$ for all $i \in \mathcal{N}$. When Assumption 2 is satisfied and $k \in [0, \frac{1}{3}]$, a unique symmetric CPNE exists, in which all contestants exert an effort $x^*(k)$, with*

$$x^*(k) = \frac{N-1}{N^2}v - \frac{(N-1)(N-2)}{N^3}kv.$$

A contestant's equilibrium effort strictly decreases with k, i.e., $dx^*(k)/dk < 0$.

Proposition 4 states that with three or more homogeneous contestants, loss aversion always weakens effort incentives: A contestant's equilibrium effort $x^*(k)$ strictly decreases with k. The contrast with the observation obtained in the symmetric two-player case reveals the nuance caused by additional players. To see this, recall that the impact of loss aversion is defused in the two-player case because each contestant wins with a probability of $1/2$. In a multi-player case, *every* contestant is technically an "underdog" despite the symmetry: One wins with a probability of $1/N$ and behaves as if he were competing against an opponent who bids $(N-1)$ times as much as he does. The uncertainty-reducing effect arises, which compels all contestants to decrease their efforts, as the underdog does in the asymmetric two-player contest. However, the competition effect differs. Each contestant i responds to $\sum_{j \neq i} x_j$, which amounts to $(N-1)x_i$ for a symmetric effort profile. The competition effect encourages each contestant to step up efforts, as $\sum_{j \neq i} x_j$ decreases due to the uncertainty-reducing effect. However, Proposition 4 shows that the (indirect) competition effect only partly offsets the (direct) uncertainty-reducing effect and cannot reverse it.

We then proceed to the more complex case of asymmetric players. Technically, the equilibrium analysis is complicated enormously by the fact that a player may choose to stay inactive—i.e., exerting zero effort—in the contest, in which case a corner equilibrium arises and equilibrium conditions are rendered elusive. A closed-form solution to the equilibrium with expectation-based loss aversion—i.e., $k > 0$—is in general unavailable: A system of nonlinear first-order conditions emerges, which rules out a handy solution. However, we verify that contestants' equilibrium efforts are well behaved, which allows us to conduct comparative statics of k at the margin of $k = 0$. The observations suffice to demonstrate the subtle incentive effects imposed by loss aversion in the extended contest setting.

Denote by m the number of active players when contestants have standard preferences (i.e., $k = 0$). The following proposition can be obtained.

Proposition 5 ***(Impact of reference-dependent preferences on incentives in multi-player contests)*** *Suppose that $N \geq 3$, Assumption 2 is satisfied, and $k \in [0, \frac{1}{3}]$. Then $x_i^*(k)$ is differentiable almost everywhere. If $v_1 \geq v_2 \geq \ldots \geq v_m$, with strict inequality holding for at least one, then one of the following three possibilities regarding $\frac{d\boldsymbol{x}^*}{dk}\big|_{k=0} \equiv \left(\frac{dx_1^*}{dk}\big|_{k=0}, \ldots, \frac{dx_N^*}{dk}\big|_{k=0}\right)$ must hold:*

a. $\frac{dx_i^*}{dk}\big|_{k=0} \leq 0$ for all $i \in \mathcal{N}$;

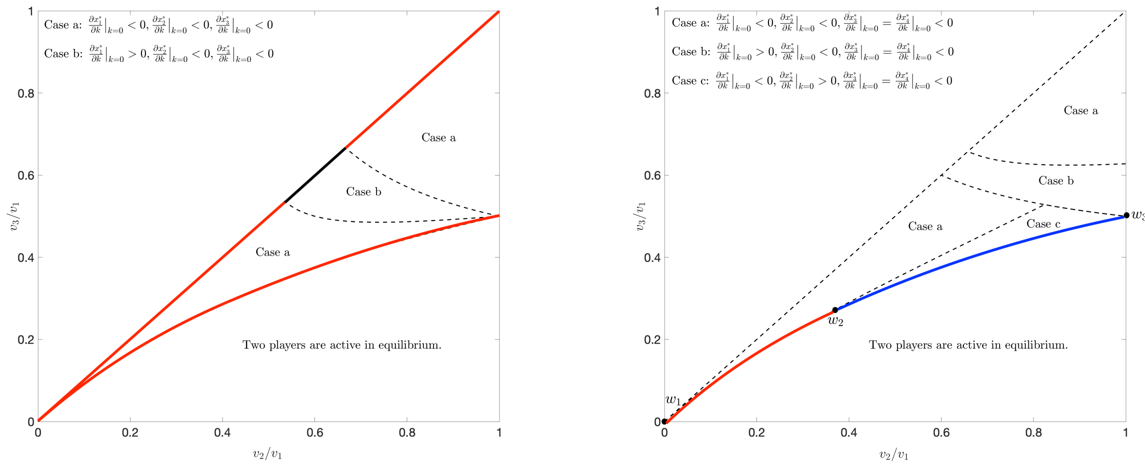
b. There exists a cutoff $\tau_x \in \{1, \ldots, m-1\}$ such that $\frac{dx_i^*}{dk}\big|_{k=0} > 0$ for $i \in \{1, \ldots, \tau_x\}$ and $\frac{dx_i^*}{dk}\big|_{k=0} \leq 0$ otherwise;

c. There exists a cutoff $\widehat{\tau}_x \in \{2, \ldots, m-1\}$ such that $\frac{dx_1^*}{dk}\big|_{k=0} < 0$, $\frac{dx_i^*}{dk}\big|_{k=0} \geq 0$ for $i \in \{2, \ldots, \widehat{\tau}_x\}$, and $\frac{dx_i^*}{dk}\big|_{k=0} \leq 0$ otherwise.

The incentive effect of loss aversion sensitively depends on the profile of contestants' prize valuations and the number of contestants. Despite the complexity caused by the heterogeneity, Proposition 5 states that three patterns are possible. In case (a), all contestants decrease their efforts. Equilibrium efforts bifurcate in case (b), in that strong contestants step up their efforts, whereas weaker contestants do the opposite. Case (c) instead depicts a nonmonotone pattern: A set of middle-ranked contestants—i.e., $\{2, \ldots, \widehat{\tau}_x\}$—increase their bids, while the rest are all discouraged, including the top contender (i.e., contestant 1).

We illustrate and elaborate on these cases in Figure 3. The left panel [Figure 3(a)] depicts a scenario of three contestants, while the right panel [Figure 3(b)] represents one of four contestants.

Three-contestant Scenario: Figure 3(a) In Figure 3(a), the horizontal axis represents the ratio v_2/v_1 and the vertical axis v_3/v_1, with both ranging from 0 to 1. The area below the



(a) Three players: $v_1 \geq v_2 \geq v_3$

(b) Four players: $v_1 \geq v_2 \geq v_3 = v_4$

Figure 3: Impact of Reference-dependent Preferences on Player Incentives.

diagonal encompasses all parameterizations relevant to our model, i.e., with $v_1 \geq v_2 \geq v_3$. As mentioned above, one may choose to stay inactive when three or more contestants are involved. The bottom-right region of the figure depicts such a situation. We focus on the region between its boundary and the diagonal, in which all three contestants remain active.

For clarity, we consider a scenario of $v_1 \geq v_2 = v_3$, which is represented by the diagonal in Figure 3(a). This scenario differs from the two-player asymmetric contest only by adding an additional weak contestant, but suffices to highlight the subtlety involved in the extended scenario. The lower portion of the diagonal depicts large asymmetry, in that v_2 and v_3 are relatively small compared with v_1. Case (a) of Proposition 5 takes place, in which all contestants decrease their efforts. This observation is parallel to the finding of Proposition 3(ii) in the highly asymmetric two-player contest, in which case both strong and weak contestants reduce their bids. In the three-player scenario, contestant 1 faces two weak opponents and is a dominating player in the competition. The uncertainty-reducing effect leads him to step up his effort and his two opponents to decrease theirs. However, because of the competition effect, the concession of his (weak) opponents tempts contestant 1 to slack off, which more than offsets the uncertainty-reducing effect, as in the two-player setting.

Along the middle portion of the diagonal, the prize valuations of contestants 2 and 3 are closer to v_1, which gives rise to case (b), with equilibrium efforts bifurcating between the strong and the weak. An analogy can also be drawn between this observation and that of Proposition 3(ii) for mildly asymmetric two-player contest: The strong contestant increases his effort, while the weak decreases it. In this case, contestant 1's advantage is limited but remains the favorite, i.e., with $p_1(x_1, \boldsymbol{x}_{-1}) > 1/2$; he steps up his effort to increase his winning odds, to reduce the uncertainty he faces. The competition effect is insufficient to reverse the uncertainty-reducing effect.

Case (a) is revived in the upper portion of the diagonal, in which case v_2 and v_3 are closer to v_1, so a more even race is in place. The observation here, however, is driven by a different force than that in the case of large asymmetry with small v_2 and v_3, i.e., the lower portion of the diagonal. Analogous to the situation depicted in Proposition 4 (symmetric multi-player contests), contestant 1 is unable to dominate the competition and should be viewed, technically, as an underdog despite his advantage over each individual opponent: His effort falls below $x_2^* + x_3^*$, and he wins with a probability less than 1/2. The uncertainty-reducing effect, which outweighs the competition effect, leads all contestants to reduce their efforts.

By Figure 3, case (c) cannot arise in any three-player contest. This observation can be proved and is formally presented in the following corollary.

Corollary 1 *Suppose that $N = 3$ and Assumption 2 is satisfied. Then either case (a) or case (b) holds; case (c) never occurs.*

Four-contestant scenario: Figure 3(b) We are now ready to incorporate an additional treatment. We assume $v_1 \geq v_2 \geq v_3 = v_4$ to fit the four-player scenario into the two-dimensional diagram [Figure 3(b)]. Similar to Figure 3(a), the area below the diagonal is a full collection of the parameterizations relevant to the setting. Contestants 3 and 4 are homogeneous, so they must employ the same strategy in the equilibrium. As a result, the set of parameterizations that cause contestant 3 to stay inactive in the three-contestant scenario are identical to those that lead both contestants 3 and 4 to drop out of the competition in the current scenario. To put this alternatively, the region in which contestant 3 stays inactive in Figure 3(a) coincides with the one in Figure 3(b), in which both contestants 3 and 4 drop out, bounded from above by the curve in Figure 3(b) that connects points w_1 (origin), w_2, and w_3: In this case, contestants 3 and 4 are excessively weak relative to their peers. Again, we focus on the region between this curve and the diagonal, in which case all four contestants remain active and exert positive efforts.

Comparing Figure 3(b) with Figure 3(a), it is straightforward to see that the case (c) of Proposition 5 is now possible. We now elaborate on the logic to demonstrate how the additional player could make a difference. For clarity and simplicity, we focus on the curve that connects points w_1, w_2, and w_3, which depicts the boundary case in which contestants 3 and 4 marginally prefer to stay active in the competition.

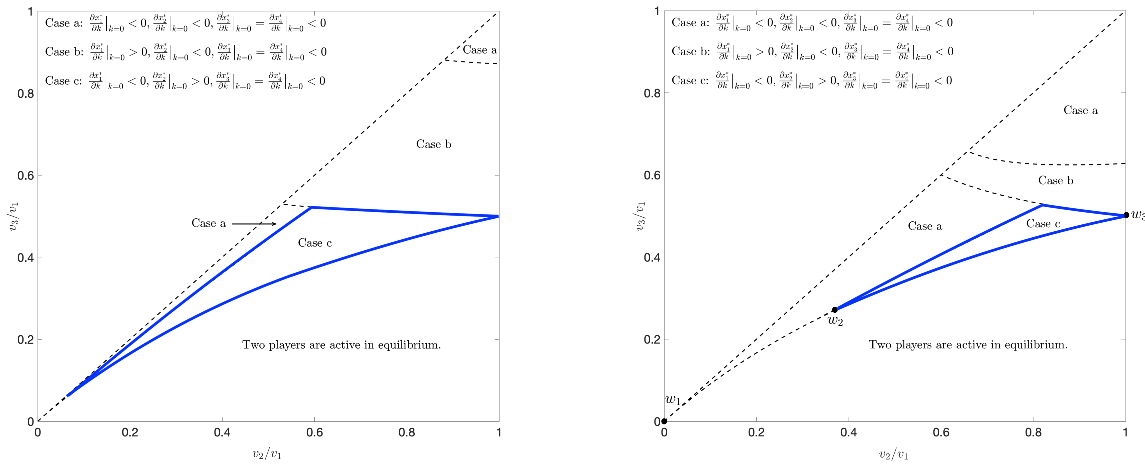
Compare Figure 3(b) with Figure 3(a). Along the portion between w_1 and w_2, case (a) continues to apply, in which all contestants reduce their efforts. However, case (c) arises for the portion that departs from w_2. It is intuitive to conclude that the ascent of case (c) requires weak contestants 3 and 4 but a relatively stronger contestant 2.

To interpret the nuance, we consider the neighborhoods of the two extremes, points w_1 and w_3. In the neighborhood of w_1, Contestants 3 and 4 are weak, but contestant 2 is close to them, with contestant 1 dominating the competition. In this situation, contestant 2 is not expected to behave much differently from contestants 3 and 4. The role played by loss aversion does not qualitatively depart from those in the three-player scenario and the highly asymmetric two-player contest: Uncertainty-reducing effect causes bifurcation between the strong and the weak, while the competition effect leads the strong to slack off, which leads to $(dx_i^*/dk)\big|_{k=0} \leq 0$ for all $i \in \mathcal{N}$.

Consider the other extreme, the neighborhood of point w_3: Contestant 2 is close to contestant 1, and contestants 3 and 4 are substantially weaker than both contestants 1 and 2. Case (a) would arise without contestant 4, while case (c) arises with the additional player. Note that contestants 3 and 4 are marginalized in this boundary case, so their winning probability is close to zero; contestant 1 is expected to win the contest with a probability marginally above 1/2, while contestant 2 does with a probability marginally below 1/2. Compared to the counterpart in the three-player scenario, the winning odds of contestants 1 and 2 are barely affected by the addition of contestant 4, while those of contestant 3 would sharply reduce, i.e., by approximately half. Further recall that the gain-loss utility, defined as $\widetilde{U}_i(x_i, \boldsymbol{x}_{-i}) := -kp_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i$, is concave in x_i, and the uncertainty-reducing effect is particularly more intense when $p_i(x_i, \boldsymbol{x}_{-i})$ is closer to zero. This implies that the uncertainty-reducing effect for contestant 3 strengthens substantially when contestant 4 joins the competition and halves his winning odds. In contrast, the uncertainty-reducing effect for contestants 1 and 2 is weak regardless—because p_1 and p_2 are close to $1/2-$and would remain nearly the same with the addition of contestant 4. The significantly strengthened direct effect on contestant 3—doubled by that on contestant 4—implies that contestant 1 or 2 would expect a relatively more significant reduction in their opponents' aggregate effort, which in turn triggers a more significant competition effect. By the standard argument à la Dixit (1987), contestant 1 would further reduce his effort as the favorite, while contestant 2 would do the opposite as an underdog. The relatively more significant competition effect on contestant 2 more than offsets his (weak) uncertainty-reducing effect: Case (c) thus emerges.

In conclusion, the addition of another weak contestant in the neighborhood of w_3 strengthens the uncertainty-reducing effect on a weak contestant and also amplifies the competition effect on the stronger contestants (1 and 2), which ultimately leads to the nonmonotone pattern described by case (c). Comparing the two extremes, w_1 and w_3, it is intuitive to infer that case (c) is more likely to occur when the middle contestant possesses a larger advantage against those at the bottom. We then observe that case (c) emerges when the boundary curve extends beyond point w_2.

The logic expounds the role of additional players. By the same token, it would be natural



(a) Ten players: $v_1 \geq v_2 \geq v_3 = v_4 \ldots = v_{10}$ (b) Four players: $v_1 \geq v_2 \geq v_3 = v_4$

Figure 4: Impact of Reference-dependent Preferences on Player Incentives.

to infer that case (c) is more likely when more (weak) players are included in the competition. Figure 4 affirms this conjecture: Figure 4(a) depicts a situation in which six more contestants identical to contestants 3 and 4 are included in the competition, with $v_1 \geq v_2 \geq v_3 = v_4 = \ldots = v_{10}$; the area for case (c) enlarges, compared with Figure 4(b).

3.3 Equilibrium Outcome

Despite the mixed responses to loss aversion in terms of individual efforts, we can obtain unambiguous predictions regarding its impact on equilibrium outcomes, i.e., the set of active contestants, total effort, and equilibrium winning probability distribution.

For notational convenience, denote by $\mathcal{M}(k)$ the set of active players under parameter k. Proposition 1 implies that a contestant must have resigned before a stronger one does, and thus $\mathcal{M}(k) = \Big\{1, \ldots, \big|\mathcal{M}(k)\big|\Big\}$.

Proposition 6 ***(Impact of reference-dependent preferences on number of active contestants)*** *Suppose that $N \geq 3$, $k \in [0, \frac{1}{3}]$, and Assumption 2 is satisfied. Then $\mathcal{M}(k) \subseteq \mathcal{M}(0)$ and thus $\big|\mathcal{M}(k)\big| \leq \big|\mathcal{M}(0)\big|$.*

Proposition 6 states that whenever a pure-strategy equilibrium exists, i.e., $k \in [0, \frac{1}{3}]$, expectation-based loss aversion always leads to a smaller set of active contestants. That is, weak contestants are more likely to drop out of the competition when they are subject to this behavioral bias. The intuition is straightforward. Recall that $\Big|\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})/\partial x_i\Big|$ is decreasing when x_i falls below $\sum_{j \neq i} x_j$, which implies that the uncertainty-reducing effect discourages relatively weaker contestants more significantly.

Proposition 5 shows that three possible patterns can be observed in response to loss aversion for individual equilibrium efforts. However, its impact on total effort—i.e., $\sum_{i=1}^{N} x_i^*$—is clear-cut.

Proposition 7 ***(Impact of reference-dependent preferences on total effort)*** *Suppose that $N \geq 2$ and Assumption 2 is satisfied. The following statements hold:*

i. If $\left|\mathcal{M}(0)\right| = 2$ and $v_1 = v_2$, then $\sum_{i=1}^{N} \frac{dx_i^*}{dk}\big|_{k=0} = 0$;

ii. Otherwise, $\sum_{i=1}^{N} \frac{dx_i^*}{dk}\big|_{k=0} < 0$.

Consistent with Proposition 6, Proposition 7 affirms the overall discouraging role played by expectation-based loss aversion. Total effort decreases except for the knife-edge case in which two homogeneous top contenders remain active in the competition: The game reduces to the symmetric two-player contest in which the impact of loss aversion disappears. In case (b) of Proposition 5, a dominant contestant can be encouraged by the uncertainty-reducing effect to further step up his bid; the incentive, however, is always eroded by the ensuing competition effect caused by the concession of his weaker opponents. In case (c), middle contestants can be further motivated by the competition effect; the second-order effect, however, is insufficient to outweigh the across-the-board decrease in the efforts of the others.

We next investigate the impact on equilibrium winning probability distribution. Note that $\mathcal{M}(k) = \mathcal{M}(0)$ for small k. The following result can be obtained.

Proposition 8 ***(Bifurcating equilibrium winning odds)*** *Suppose that $N \geq 2$ and Assumption 2 is satisfied. The following statements hold:*

i. If $v_1 = \ldots = v_{\mathcal{M}(0)} =: v$, then $p_i^* = \frac{1}{\mathcal{M}(0)}$ for all $i \in \mathcal{M}(0)$ and $k \in [0, \frac{1}{3}]$;

ii. If $v_1 \geq v_2 \geq \ldots \geq v_{\mathcal{M}(0)}$, with strict inequality holding for at least one, then there exists a cutoff $\tau_p \in \{1, \ldots, N-1\}$ such that $\frac{dp_i^*}{dk}\big|_{k=0} > 0$ for $i \leq \tau_p$ and $\frac{dp_i^*}{dk}\big|_{k=0} \leq 0$ for $i > \tau_p$.

Despite the mixed patterns of changes in effort incentives $(dx_i^*/dk)\big|_{k=0}$, loss aversion causes winning probabilities to bifurcate between strong and weak contestants. A more elitist distribution pattern results, as winning odds are increasingly concentrated on the top contestants.

4 Discussion: Alternative Equilibrium Concept

In this section, we consider another popular equilibrium concept in the literature to verify the robustness of our main results. Kőszegi and Rabin (2006) propose the notion of *personal equilibrium* (PE) to depict the consistent behavior of expectation-based loss-averse individuals. Dato, Grunewald, Müller, and Strack (2017) and Dato, Grunewald, and Müller (2018) extend the concept of PE to situations that involve strategic interactions and develop the concept of personal Nash equilibrium (PNE), which is formally defined as follows.

Definition 2 ***(Personal Nash equilibrium)*** *The effort profile* $\boldsymbol{x}^{**} \equiv (x_1^{**}, \ldots, x_N^{**})$ *constitutes a personal Nash equilibrium (PNE) in pure strategy if for all* $i \in \mathcal{N}$,

$$U_i(x_i^{**}, x_i^{**}, \boldsymbol{x}_{-i}^{**}) \geq U_i(x_i, x_i^{**}, \boldsymbol{x}_{-i}^{**}), \textit{ for all } x_i \in [0, \infty).$$

The concept of PE requires that a contestant's reference point be fixed (i.e., choice-unacclimating), and not adjust to his choice of effort when taking action. A PNE further requires that all contestants be willing to follow their credible effort plan. As previously stated, the notions of PE and PNE are arguably more plausible for contexts in which outcomes are realized shortly after players take their actions, in that their expectations do not have enough time to adapt to actual decisions and can be considered exogenous.

In contrast to the concept of CPNE, contestants with *fixed* expectations under PNE are attached to the amount of effort they expected to sink, and thus there may exist multiple plans a contestant is willing to implement. Multiple equilibria may often arise. To address the issue of multiple equilibria, Kőszegi and Rabin (2006) argue that agents should be expected to choose their most preferred PE, which gives rise to the concept of *preferred personal equilibrium* (PPE) and *preferred personal Nash equilibrium* (PPNE), PPE's game-theoretic variant (Dato, Grunewald, Müller, and Strack, 2017; Dato, Grunewald, and Müller, 2018).

Following Dato, Grunewald, and Müller (2018), denote by $\Theta_i(\boldsymbol{x}_{-i})$ the set of pure-strategy PEs of contestant i for a given effort profile of his opponents, $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. PPNE is formally defined as follows.

Definition 3 ***(Preferred personal Nash equilibrium)*** *An effort profile* $\boldsymbol{x}^{**} \equiv (x_1^{**}, \ldots, x_N^{**})$ *constitutes a preferred personal Nash equilibrium (PPNE) in pure strategy if for all* $i \in \mathcal{N}$,

$$U_i(x_i^{**}, x_i^{**}, \boldsymbol{x}_{-i}^{**}) \geq U_i(x_i, x_i, \boldsymbol{x}_{-i}^{**}), \textit{ for all } x_i \in \Theta_i(\boldsymbol{x}_{-i}^{**}).$$

Recall that a loss-averse contestant's utility function is given by

$$U_i(x_i, \hat{x}_i, \boldsymbol{x}_{-i}) = p_i(x_i, \boldsymbol{x}_{-i}) \times \Big\{v_i + \eta \left[1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})\right] \times \mu(v_i)\Big\} + \left[1 - p_i(x_i, \boldsymbol{x}_{-i})\right] \times \big\{0 + \eta p_i(\hat{x}_i, \boldsymbol{x}_{-i}) \times \mu(-v_i)\big\} - x_i + \eta\mu(\hat{x}_i - x_i),$$

where the parameter $\eta \geq 0$ is the weight a contestant attaches to his gain-loss utility relative to his material utility; and $\mu(\cdot)$ is the universal psychological gain-loss utility and is defined as the following:

$$\mu(c) = \begin{cases} c & \text{if} \quad c \geq 0, \\ \lambda c & \text{if} \quad c < 0. \end{cases}$$

Further recall that k is defined as $k := \eta(\lambda - 1)$. For a generic contest game, we first obtain the following.

Theorem 2 ***(Existence and uniqueness of PPNE with moderate loss aversion)*** *There exists a unique pure-strategy PPNE in the contest game if Assumption 1 is satisfied and* $k \in [0, \frac{1}{3}]$.

In parallel to Theorem 1, Theorem 2 establishes the existence and uniqueness of PPNEs for moderate loss aversion.[16] It remains elusive to what extent the prediction under PPNE differs from its counterpart under CPNE.

Theorem 3 *Suppose that Assumption 1 is satisfied and fix* $\lambda > 1$. *Then there exists a threshold* $\tilde{\eta} \in \left(0, \frac{1}{3(\lambda-1)}\right)$ *such that for all* $\eta < \tilde{\eta}$, *the unique CPNE of the contest game coincides with the unique PPNE.*

Theorem 3 states that PPNE is equivalent to CPNE when η is sufficiently small, which implies that the contestant's concern about his gain-loss utility remains tempered: The prediction obtained under Section 3 would remain intact in this case even if PPNE were adopted as the solution concept.

When η exceeds the threshold, the prediction under PPNE may depart from that under CPNE. Next, we provide two examples to show that the main results under Section 3 are qualitatively robust under the alternative equilibrium concept.

4.1 PPNE and CPNE in Two-player Contests

We first consider a two-player contest, as in Section 3.1. Suppose that $f_i(x_i) = x_i$, $N = 2$, and $v_1 \geq v_2$. Then the unique pure-strategy CPNE coincides with the unique pure-strategy

[16] Although PPE is uniquely determined in situations of individual decision-making and can be considered to be a reasonable selection criterion, the existence of PPNEs cannot always be guaranteed in general. See Dato, Grunewald, Müller, and Strack (2017) for detailed discussions.

PPNE if and only if

$$\frac{v_1}{v_2} \leq 1 + \frac{\left(\frac{1+\eta\lambda}{\eta\lambda}\right)^2 - 1}{\frac{1+\eta\lambda}{\eta\lambda} \times \frac{1-\eta+\eta\lambda}{1+\eta-\eta\lambda} + 1}.$$

Set $\lambda = 1.25$ and $\eta = 1$, and normalize $v_2 = 1$ without loss of generality. Then CPNE is the same as the PPNE when $v_1/v_2 \leq 39/25 = 1.56$.

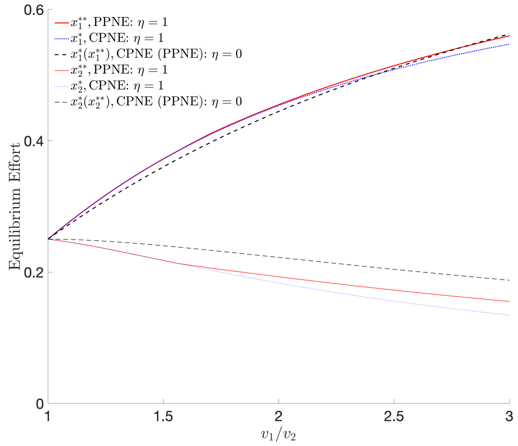
Figure 5 reports contestants' equilibrium effort profile, total effort, and the equilibrium winning probabilities of the strong contestant in the unique CPNE and PPNE under different levels of v_1/v_2, as well as the counterparts under standard preferences, i.e., $\eta = 0$. A few remarks are in order. First, by Figure 5(a), the weak contestant always exerts a lower effort in PPNE with the presence of loss aversion than he would under standard preferences. In contrast, loss aversion leads the strong contestant to raise his effort if v_1/v_2 is sufficiently small. These observations affirm the results of Proposition 3 under CPNE. Second, by Figure 5(b), the total effort of loss-averse contestants in the PPNE is always less than under standard preference, which echoes the claim of Proposition 7. Finally, by Figure 5(c), loss aversion causes the equilibrium winning odds to bifurcate between the strong contestant and the weak one in the PPNE. Specifically, the strong contestant is more likely to prevail in the competition when η increases from 0 to 1. This observation, again, is consistent with the prediction of Proposition 8 under CPNE.

4.2 PPNE and CPNE in Contests with Three or More Contestants

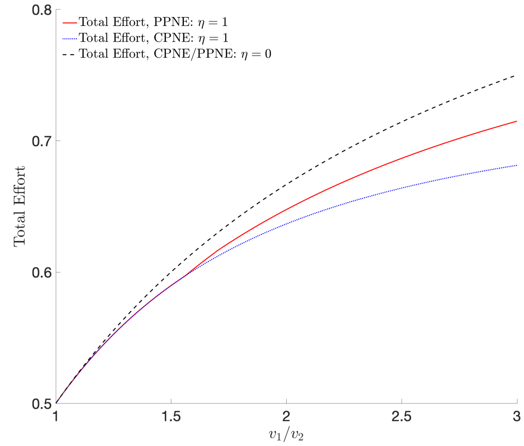
We now consider a multi-player contest, as in Section 3.2. Suppose that $f_i(x_i) = x_i$. Set $(N, \lambda) = (8, 1.2)$, and $\boldsymbol{v} \equiv (v_1, v_2, \ldots, v_8) = (2.8, 2.7, \ldots, 2.1)$. The following table reports the equilibrium winning probability distribution in the unique CPNE and PPNE when they are loss averse (i.e., $\eta = 1$), as well as that under standard preferences (i.e., $\eta = 0$).

η	Equilibrium concept	p_1^*	p_2^*	p_3^*	p_4^*	p_5^*	p_6^*	p_7^*	p_8^*	Total effort
0	NE/CPNE/PPNE	0.2396	0.2115	0.1811	0.1484	0.1129	0.0743	0.0322	0	2.1291
1	CPNE	0.2879	0.2486	0.2039	0.1522	0.0910	0.0164	0	0	1.8247
1	PPNE	0.2479	0.2177	0.1851	0.1495	0.1106	0.0680	0.0211	0	1.9619

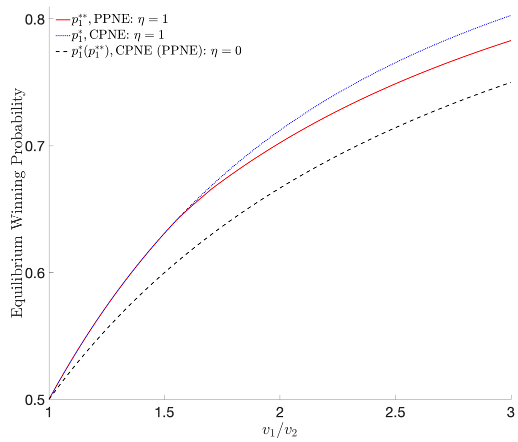
CPNE and PPNE differ for $\eta = 1$. However, the main prediction obtained in Propositions 7 and 8 under CPNE remain qualitatively intact under PPNE. Specifically, the equilibrium winning distributions under both CPNE and PPNE become more dispersed when contestants are loss averse, as compared with that under standard preferences (i.e., $\eta = 0$). In particular, the strongest (weakest) four contestants have higher (lower) winning odds when



(a) Equilibrium Effort



(b) Total Effort



(c) Equilibrium Winning Probability

Figure 5: PPNE vs. CPNE: $(N, v_2, \lambda) = (2, 1, 1.25)$.

$\eta = 1$ than when $\eta = 0$, regardless of the equilibrium concept. Moreover, the total effort of the contest decreases when loss aversion is in place: Under CPNE, it drops from 2.1291 to 1.8247, while under PPNE, it reduces to 1.9619.

5 Concluding Remark

This paper explores the equilibrium interplay in contests with loss averse contestants à la Kőszegi and Rabin (2006, 2007). We first establish the existence and uniqueness of CPNE in a contest game. We then investigate the incentive effects of loss aversion, as well as its impact on the equilibrium winning probability distribution. We demonstrate that loss aversion yields subtle effects on contestants' behavior: It catalyzes a (direct) uncertainty-reducing effect, which further triggers an (indirect) competition effect. The tension between the two effects could lead contestants to either increase or decrease their efforts. Despite the ambiguous impact of expectation-based loss aversion on incentives, we show that its impact on the set of active players, total effort, and equilibrium winning probability distribution is clear-cut. Finally, we consider another popularly studied equilibrium concept (PPNE) and show that our main predictions continue to hold qualitatively.

Our paper is an early foray into the implications of expectation-based loss aversion in contests. Large room remains for future studies. It would be interesting to investigate loss-averse players' incentives/strategies in other competitive settings, such as all-pay auctions[17] and penny auctions (Hinnosaar, 2016) when bidders are ex ante heterogeneous. Recently, Goette, Graeber, Kellogg, and Sprenger (2019) examine the role of heterogeneity in gain-loss attitude in identifying models of expectation-based reference dependence. The study is placed in an individual-decision setting. It is important to examine the competition between contestants who differ in their levels of loss aversion. This study abstracts away contestants' decision to participate in a competition. It would be intriguing to examine loss-averse contestants' incentives to enter a contest when it entails an upfront cost. Alternatively, Azmat and Möller (2009, 2018) and Morgan, Sisak, and Várdy (2018) study contestants' self-selection into different contests. The question warrants reexamination when loss aversion is embedded in contestants' preferences. The result would enlighten a contest designer who sets contest rules to attract participation when she faces competitions from other contests. We leave the exploration of these possibilities to future research.

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[17]See Fu, Wu, and Zhu (2019) for further investigations along this direction.

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Appendix: Proofs

Proof of Theorem 1

Proof. Define $y_i := f_i(x_i)$ and $\boldsymbol{y} := (y_1, \ldots, y_N)$. Moreover, denote the inverse function of $f_i(\cdot)$ by $\phi_i(\cdot) := f_i^{-1}(\cdot)$. Then the expected utility of contestant i in expression (3) can be rewritten as

$$\widehat{\pi}_i(\boldsymbol{y}) = \frac{y_i}{\sum_{j=1}^N y_j} v_i - k\frac{y_i}{\sum_{j=1}^N y_j} \times \left(1 - \frac{y_i}{\sum_{j=1}^N y_j}\right) v_i - \phi_i(y_i).$$

It can be verified that $\widehat{\pi}_i(\cdot)$ is strictly concave in $y_i > 0$ if $k \leq \frac{1}{3}$. Therefore, if $\left.\frac{\partial \widehat{\pi}_i(\boldsymbol{y})}{\partial y_i}\right|_{y_i=0} \leq 0$, or equivalently, $\phi_i'(0)s \geq (1-k)v_i$, then $y_i = 0$. Otherwise, y_i is strictly positive and solves $\left.\frac{\partial \widehat{\pi}_i(\boldsymbol{y})}{\partial y_i}\right|_{y_i=0} = 0$. Carrying out the algebra, we have

$$\frac{s - y_i}{s^2} \times \left(1 - k + 2k\frac{y_i}{s}\right) = \frac{1}{v_i} \times \phi_i'(y_i), \tag{5}$$

where s is defined as $s := \sum_{j=1}^N y_j$.

Note that $s = 0$ cannot arise in equilibrium. Otherwise, $x_1 = \ldots = x_N = 0$ and a contestant has strict incentive to deviate by increasing his effort $x_i = 0$ to a sufficiently small positive amount. For all $s > 0$, let us define

$$g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \leq \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left(1 - k + 2k\frac{y_i}{s}\right) = \frac{1}{v_i}\phi_i'(y_i) & \text{otherwise,} \end{cases} \tag{6}$$

Next, we show that $g_i(s)$ is well defined. If $(1-k)v_i > \phi_i'(0)s$, then at $y_i = 0$, $\frac{1}{s}(1-k) > \frac{1}{v_i}\phi_i'(0)$; and with $y_i = s$, $0 < \frac{1}{v_i}g_i'(s)$. Moreover, the left-hand side of Equation (5) is strictly decreasing in y_i if $k \leq \frac{1}{3}$ and the right-hand side is weakly increasing in y_i under Assumption 1. Similarly, the left-hand side of Equation (5) is quadratic and is inverse U-shaped in y_i if $\frac{1}{3} < k \leq \frac{1}{2}$, and the right-hand side is weakly convex and weakly increasing in y_i under Assumption A1. Therefore, the unique solution in the interval $(0, s)$ is guaranteed in both cases.

From the above analysis, it is evident that the effort profile $\boldsymbol{x} \equiv (x_1^*, \ldots, x_N^*)$ constitutes a CPNE if and only if $\sum_{i=1}^N g_i(s) = s$, or equivalently, $\chi(s) := \sum_{i=1}^N \frac{g_i(s)}{s} - 1 = 0$. Define $\rho_i(s) := \frac{g_i(s)}{s}$. Then Equation (6) indicates that $\rho_i(s) = 0$ for $s \geq \frac{(1-k)v_i}{\phi_i'(0)}$. For $s < \frac{(1-k)v_i}{\phi_i'(0)}$, it follows from Equation (5) that

$$(1 - \rho_i) \times (1 - k + 2k\rho_i)\, v_i - s \times \phi_i'(\rho_i s) = 0, \tag{7}$$

which in turn implies that

$$\rho_i'(s) = -\frac{\phi_i'(\rho_i s) + \rho_i s \times \phi_i''(\rho_i s)}{(1 - 3k + 4k\rho_i) v_i + s^2 \times \phi_i''(\rho_i s)}, \tag{8}$$

from the implicit function theorem. Because $\phi_i' > 0$ and $\phi_i'' \geq 0$, the numerator on the right-hand side of the above equation is strictly positive. Next, we show that the sign of the denominator is strictly positive.

Clearly, we have

$$(1 - 3k + 4k\rho_i) v_i + s^2 \times \phi_i''(\rho_i s) \geq 4k\rho_i v_i + s^2 \times \phi_i''(\rho_i s) > 0, \tag{9}$$

where the first inequality follows from $k \leq \frac{1}{3}$. To complete the proof, it remains to show that $\chi(s) := \sum_{i=1}^N \rho_i(s) - 1 = 0$ has a unique positive solution for the case $k \leq \frac{1}{3}$.

First, note that $\chi(s)$ is strictly decreasing in s for $s \in (0, \frac{(1-k)v_1}{\phi_i'(0)})$, and is constant for $s \geq \frac{(1-k)v_1}{\phi_i'(0)}$. It is straightforward to see that $\rho_i(s)$ is continuous in s and thus $\chi(s)$ is continuous in s. Second, note that

$$\chi\left(\frac{(1-k)v_1}{\phi_i'(0)}\right) = -1.$$

Moreover, it follow from Equation (7) that $\lim_{s\downarrow 0} \rho_i(s) = 1$, and

$$\lim_{s\searrow 0} \chi(s) = N - 1 > 0.$$

Hence, the unique positive solution to $\chi(s) = 0$ is guaranteed. ■

Proof of Proposition 1

Proof. Note that $f_i(x_i) = x_i$ implies instantly that $y_i = x_i$, and thus $s := \sum_{i=1}^N y_i = \sum_{i=1}^N x_i$. Fixing $k \in (0, \frac{1}{2}]$, the function $g_i(s)$ defined in Equation (6) can be simplified as

$$g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \leq s, \\ \frac{\sqrt{(1-3k)^2 s^2 + 8ks^2\left(1-k-\frac{s}{v_i}\right)} - (1-3k)s}{4k} & \text{otherwise,} \end{cases}$$

which is exactly the expression as in (4). It follows from the proof of Theorem 1 that in the unique CPNE, we must have that $x_i^* = g_i(s)$, where $s > 0$ is the unique solution to $\sum_{i=1}^N g_i(s) = s$. This completes the proof. ■

Proof of Proposition 2

Proof. It follows from the first-order conditions $\frac{\widehat{U}_1(x_1,x_2^*)}{\partial x_1}\Big|_{x_1=x_1^*} = 0$ and $\frac{\widehat{U}_2(x_2,x_1^*)}{\partial x_2}\Big|_{x_2=x_2^*} = 0$ that

$$\frac{x_2^*}{(x_1^*+x_2^*)^2}v_1 - \frac{x_2^*(x_2^*-x_1^*)}{(x_1^*+x_2^*)^3}kv_1 = 1, \tag{10}$$

and

$$\frac{x_1^*}{(x_1^*+x_2^*)^2}v_2 - \frac{x_1^*(x_1^*-x_2^*)}{(x_1^*+x_2^*)^3}kv_2 = 1. \tag{11}$$

Let $\theta := \frac{x_1^*}{x_2^*}$. Then Equations (10) and (11) can be rewritten as

$$\frac{1}{1+\theta}v_1 - \frac{1-\theta}{(1+\theta)^2}kv_1 = x_1^* + x_2^*,$$

and

$$\frac{\theta}{1+\theta}v_2 - \frac{\theta(\theta-1)}{(1+\theta)^2}kv_2 = x_1^* + x_2^*.$$

Combining the above two equations yields

$$\frac{1}{1+\theta}v_1 - \frac{1-\theta}{(1+\theta)^2}kv_1 = \frac{\theta}{1+\theta}v_2 - \frac{\theta(\theta-1)}{(1+\theta)^2}kv_2,$$

which is equivalent to

$$(1-k)\theta^2 - \left(\frac{v_1}{v_2}-1\right) \times (1+k)\theta - \frac{v_1}{v_2}(1-k) = 0. \tag{12}$$

Solving for θ, we have that

$$\theta = \frac{1}{2}\left[\left(\frac{v_1}{v_2}-1\right)\frac{1+k}{1-k} + \sqrt{\left(\frac{v_1}{v_2}-1\right)^2\left(\frac{1+k}{1-k}\right)^2 + \frac{4v_1}{v_2}}\right]. \tag{13}$$

Substituting (13) into (10) and (11), we can solve for $x_1^*(k)$ and $x_2^*(k)$ as the following:

$$x_1^*(k) = \frac{\theta}{(1+\theta)^2}v_1 - \frac{\theta(1-\theta)}{(1+\theta)^3}kv_1, \tag{14}$$

and

$$x_2^*(k) = \frac{1}{(1+\theta)^2}v_1 - \frac{1-\theta}{(1+\theta)^3}kv_1. \tag{15}$$

This completes the proof. ■

Proof of Proposition 3

Proof. For $v_1 = v_2 =: v$, it is straightforward to verify that $\theta = 1$ from Equation (13) and hence $x_1^*(k) = x_2^*(k) = \frac{1}{4}v$; and it remains to prove the result for the case $v_1 > v_2$. For notational convenience, define $\ell := v_1/v_2 > 1$; and we add k into θ to emphasize the fact that θ depends on k. It follows from Equation (12) and the implicit function theorem that

$$\frac{d\theta(k)}{dk} = \frac{\theta^2 + (\ell-1)\theta - \ell}{2\theta(1-k) - (\ell-1)(1+k)}.$$

The above equation, together with the fact that $\theta(0) = \ell$ from Equation (13), implies that

$$\left.\frac{d\theta(k)}{dk}\right|_{k=0} = \frac{\ell^2 + (\ell-1)\ell - \ell}{2\ell - (\ell-1)} = \frac{2\ell(\ell-1)}{1+\ell} > 0. \tag{16}$$

Differentiating $x_1^*(k)$ in (14) with respect to k, we have that

$$\frac{dx_1^*(k)}{dk} = \frac{1-\theta}{(1+\theta)^3} \times \frac{d\theta}{dk} \times v_1 - \frac{\theta^2 - 4\theta + 1}{(1+\theta)^4} \times \frac{d\theta}{dk} \times kv_1 - \frac{\theta(1-\theta)}{(1+\theta)^3} v_1.$$

Note that $\theta(0) = \ell$ from (13); together with (16), we have that

$$\left.\frac{dx_1^*(k)}{dk}\right|_{k=0} = \frac{1-\ell}{(1+\ell)^3} \times \frac{2\ell(\ell-1)}{1+\ell} v_1 - \frac{\ell(1-\ell)}{(1+\ell)^3} v_1 = \frac{\ell(\ell-1)(3-\ell)}{(1+\ell)^4} v_1,$$

which in turn implies that

$$\left.\frac{dx_1^*(k)}{dk}\right|_{k=0} \gtreqless 0 \Leftrightarrow \ell \lesseqgtr 3.$$

Next, we show that $\frac{dx_2^*(k)}{dk} < 0$ for all $\ell > 1$. Differentiating $x_2^*(k)$ in (15) with respect to k yields

$$\frac{dx_2^*(k)}{dk} = \frac{1}{\theta(k)} \times \frac{dx_1^*(k)}{dk} - \frac{1}{\big[\theta(k)\big]^2} \times \frac{d\theta(k)}{dk} \times x_1^*(k).$$

Recall that $\theta(0) = \ell$ from (13). In addition, $x_1^*(0) = \frac{\ell}{(1+\ell)^2} v_1$ from (14). Therefore, we have that

$$\left.\frac{dx_2^*(k)}{dk}\right|_{k=0} = \frac{1}{\ell} \times \frac{\ell(\ell-1)(3-\ell)}{(1+\ell)^4} v_1 - \frac{1}{\ell^2} \times \frac{2\ell(\ell-1)}{1+\ell} \times \frac{\ell}{(1+\ell)^2} v_1 = -\frac{(\ell-1)(3\ell-1)}{(1+\ell)^4} v_1 < 0.$$

This completes the proof. ■

Proof of Proposition 4

Proof. The characterization of equilibrium follows immediately from Proposition 1, and is omitted for brevity. ■

Proof of Proposition 5

Proof. The following result due to Stein (2002) fully characterizes the equilibrium for the benchmark case $k = 0$ without reference-dependent preferences.

Lemma 1 ***(Stein, 2002)*** *Suppose that $f_i(x_i) = x_i$ for all $i \in \mathcal{N}$ and $k = 0$. Then the equilibrium effort profile, $\boldsymbol{x}^*(0) \equiv \big(x_1^*(0), \ldots, x_N^*(0)\big)$, is given by*

$$x_i^*(0) = \begin{cases} s(0) - \frac{[s(0)]^2}{v_i} & \text{if} \quad i \in \{1, \ldots, m\}, \\ 0 & \text{if} \quad i \in \mathcal{N} \setminus \{1, \ldots, m\}, \end{cases} \tag{17}$$

where m is the number of active players and is given by

$$m = \max \left\{ n = 2, \ldots, N \,\middle|\, (n-1)\frac{1}{v_i} < \sum_{j=1}^{n} \frac{1}{v_j} \right\},$$

and

$$s(0) \equiv \sum_{i=1}^{N} x_i^*(0) = \frac{m-1}{\sum_{i=1}^{m} \frac{1}{v_i}}. \tag{18}$$

We first prove part (i) of the proposition. For the case $v_1 = \ldots = v_m =: v$, it is straightforward to verify that $\mathcal{M}(k) = \mathcal{M}(0)$ for all $k \in [0, \frac{1}{2}]$. Moreover, the first-order condition $\frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = 0$ for $i \in \mathcal{M}(k)$ can be rewritten as

$$\frac{s - x_i^*}{s^2} v - \frac{(s - x_i^*)(s - 2x_i^*)}{s^3} kv = 1, \text{ for all } i \in \mathcal{M}(0). \tag{19}$$

Combining contestant i's and contestant j's first-order conditions, with $i \neq j$, we can obtain

$$(x_i^* - x_j^*) \times [2k(x_i^* + x_j^*) - (3k-1)s] = 0,$$

which implies instantly that $x_i^* = x_j^*$ and $s = mx_i^*$. Plugging $s = mx_i^*$ into Equation (19), we can obtain that

$$x_i^*(k) = \frac{m-1}{m^2} v - \frac{(m-1)(m-2)}{m^3} kv, \text{ for all } i \in \mathcal{M}(0).$$

Clearly, $x_i^*(k)$ is strictly decreasing in k for all $i \in \mathcal{M}(0)$ if $m \geq 3$.

Next, we prove part (ii) of the proposition. Suppose that there exist two active contestants whose winning values are different when $k = 0$. It can be verified that $\big|\mathcal{M}(k)\big| = \big|\mathcal{M}(0)\big| \equiv m$

for a sufficiently small k. First, we have that

$$\begin{aligned}
\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} &= -2\left[p_i^*(0)\right]^2 + 3p_i^*(0) - 1 - \frac{1}{v_i} \times \left.\frac{ds}{dk}\right|_{k=0} \\
&= -2\left[p_i^*(0)\right]^2 + 3p_i^*(0) - 1 - \frac{1}{v_i} \times \left\{ s(0) - \frac{2\left[s(0)\right]^3 \times \sum_{i=1}^m \frac{1}{v_i^2}}{m-1} \right\} \\
&= -2\left[p_i^*(0)\right]^2 + 3p_i^*(0) - 1 - \left[1 - p_i^*(0)\right] \times \left\{ 1 - \frac{2\sum_{i=1}^m \frac{1}{v_i^2}}{m-1} \times \left[\frac{m-1}{\sum_{i=1}^m \frac{1}{v_i}}\right]^2 \right\} \\
&= -2\left[1 - p_i^*(0)\right] \times \left\{ 1 - \frac{(m-1)\sum_{i=1}^m \frac{1}{v_i^2}}{\left[\sum_{i=1}^m \frac{1}{v_i}\right]^2} - p_i^*(0) \right\}, \qquad (20)
\end{aligned}$$

where the first equality follows from (27); the second equality follows from (30); the third equality follows from (18) and (28). Denote $1 - \frac{(m-1)\sum_{i=1}^m \frac{1}{v_i^2}}{\left[\sum_{i=1}^m \frac{1}{v_i}\right]^2}$ by $\tilde{p}$. It is straightforward to verify that $\tilde{p} < \frac{1}{2}$ for $m \geq 3$. Moreover, we must have that $\tilde{p} > 0$. Otherwise, $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ for all $i \in \mathcal{M}(0)$ from (20), and thus we have $0 = \sum_{i=1}^N \left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = \sum_{i=1}^m \left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$, a contradiction. Equation (20) implies immediately that

$$\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0 \Leftrightarrow p_i^*(0) > \tilde{p} \equiv 1 - \frac{(m-1)\sum_{i=1}^m \frac{1}{v_i^2}}{\left[\sum_{i=1}^m \frac{1}{v_i}\right]^2}. \qquad (21)$$

Second, it follows from $x_i^*(k) = p_i^*(k) \times s(k)$ that

$$\begin{aligned}
\left.\frac{dx_i^*}{dk}\right|_{k=0} &= \left.\frac{dp_i^*}{dk}\right|_{k=0} \times s(0) + \left.\frac{ds}{dk}\right|_{k=0} \times p_i^*(0) \\
&= \left\{ -2\left[p_i^*(0)\right]^2 + 3p_i^*(0) - 1 - \frac{1}{v_i} \times \left.\frac{ds}{dk}\right|_{k=0} \right\} \times s(0) + \left.\frac{ds}{dk}\right|_{k=0} \times p_i^*(0) \\
&= -s(0)\left[1 - p_i^*(0)\right]\left[1 - 2p_i^*(0)\right] + \left.\frac{ds}{dk}\right|_{k=0} \times \left[p_i^*(0) - \frac{s(0)}{v_i}\right] \\
&= -s(0) \times \left[1 - 2p_i^*(0)\right] \times \left[\frac{\left.\frac{ds}{dk}\right|_{k=0} + s(0)}{s(0)} - p_i^*(0)\right] \\
&= -s(0) \times \left[1 - 2p_i^*(0)\right] \times \left[2\tilde{p} - p_i^*(0)\right] \qquad (22) \\
&= -s(0) \times \left[\frac{2}{v_i} \times \frac{m-1}{\sum_{i=1}^m \frac{1}{v_i}} - 1\right] \times \left\{ \frac{s(0)}{v_i} - \left(\frac{2\left[s(0)\right]^2 \times \sum_{i=1}^m \frac{1}{v_i^2}}{m-1} - 1 \right) \right\}, \qquad (23)
\end{aligned}$$

where the second equality follows from Equation (27); the fourth equality follow from Equation (28); the fifth equality follows from (18), (30), and the definition of $\tilde{p}$; and the last equality follows from (18), (28), and (30). Let us define

$$v^{\ddagger} := \frac{2(m-1)}{\sum_{i=1}^{m}\frac{1}{v_i}} > 0,$$

and

$$v^{\ddagger\ddagger} := \frac{s(0)}{\frac{2\left[s(0)\right]^2\times\sum_{i=1}^{m}\frac{1}{v_i^2}}{m-1}-1} = \frac{(m-1)\sum_{i=1}^{m}\frac{1}{v_i}}{2(m-1)\sum_{i=1}^{m}\frac{1}{(v_i)^2}-\left(\sum_{i=1}^{m}\frac{1}{v_i}\right)^2} > 0.$$

Let $\underline{v} := \min\{v^{\ddagger}, v^{\ddagger\ddagger}\}$ and $\overline{v} := \max\{v^{\ddagger}, v^{\ddagger\ddagger}\}$. Equations (22) and (23) imply instantly that

$$\begin{aligned} \left.\frac{dx_i^*}{dk}\right|_{k=0} > 0 &\Leftrightarrow \min\left\{\frac{1}{2}, 2\tilde{p}\right\} < p_i^* < \max\left\{\frac{1}{2}, 2\tilde{p}\right\} \\ &\Leftrightarrow \underline{v} < v_i < \overline{v}. \end{aligned}$$

If $\underline{v} = \overline{v}$, then $\left.\frac{dx_i^*}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{N}$. Suppose that $\underline{v} < \overline{v}$. The vector of the winning values $\boldsymbol{v} \equiv (v_1, \ldots, v_m, \ldots, v_N)$ has to belong to one of the following five cases:

Case I:

$v_1 \leq \underline{v}$ or $v_m \geq \overline{v}$. Then we have that $\left.\frac{dx_i^*}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{M}(0)$, which corresponds to possibility (a) in part (ii) of the proposition.

Case II:

$\underline{v} \leq v_m < v_1 \leq \overline{v}$. Then we have that $\left.\frac{dx_i^*}{dk}\right|_{k=0} \geq 0$ for all $i \in \mathcal{M}(0)$, with strict strict inequality holding for at least one. This case is impossible because $\left.\frac{ds}{dk}\right|_{k=0} \equiv \sum_{i=1}^{m}\left.\frac{dx_i^*}{dk}\right|_{k=0} < 0$ by Proposition 7.

Case III:

$v_m \leq \underline{v} < v_1 \leq \overline{v}$. Then there exists a cutoff of winning value above which $\left.\frac{dx_i^*}{dk}\right|_{k=0} > 0$ and below which $\left.\frac{dx_i^*}{dk}\right|_{k=0} \leq 0$, which corresponds to possibility (b) in part (ii) of the proposition.

Case IV:

$\underline{v} \leq v_m \leq \overline{v} < v_1$. This implies instantly that $p_i^* > \min\left\{\frac{1}{2}, 2\tilde{p}\right\} > \tilde{p}$, where the last strict inequality follows from $\tilde{p} \in (0, \frac{1}{2})$. Together with (21), we have that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ for $i \in \mathcal{M}(0)$. This in turn implies that $\sum_{i=1}^{N}\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = \sum_{i=1}^{m}\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$, which is a contradiction. Therefore, this case is impossible.

Case V:

$v_m \leq \underline{v} \leq \overline{v} < v_1$. If there exists no contestant whose winning value lies between $\underline{v}$ and $\overline{v}$, then $\left.\frac{dx_i^*}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{M}(0)$ and again we have possibility (a). Suppose instead there exists a contestant $t \in \{2, \ldots, m-1\}$ such that $v_t \in (\underline{v}, \overline{v})$ and $v_1 > v_t > v_m$. Next we show that we must have possibility (c). It suffices to rule out the situation that $\left.\frac{dx_1^*}{dk}\right|_{k=0} < 0$ and

$\frac{dx_2^*}{dk}\Big|_{k=0} \leq 0$. We consider the following two sub-cases:

Sub-case (1): $\tilde{p} \geq \frac{1}{4}$**.** The postulated $v_m \leq \underline{v} < v_t < \overline{v} < v_1$ implies that we have that $p_1^* \geq p_t^* \geq \min\left\{\frac{1}{2}, 2\tilde{p}\right\} = \frac{1}{2}$. Together with the fact that $p_m^* > 0$, we have that $\sum_{i=1}^{N} p_i^* \geq p_1^* + p_t^* + p_m^* > 1$, which is a contradiction.

Sub-case (2): $\tilde{p} < \frac{1}{4}$**.** Because $\frac{dx_1^*}{dk}\Big|_{k=0} < 0$ and $\frac{dx_2^*}{dk}\Big|_{k=0} \leq 0$, we must have $p_1^* \geq p_2^* \geq \max\left\{\frac{1}{2}, 2\tilde{p}\right\} = \frac{1}{2}$. Together with the fact that $p_m^* > 0$, we have that $\sum_{i=1}^{N} p_i^* \geq p_1^* + p_2^* + p_m^* > 1$, which is again a contradiction.

Therefore, we can only have three patterns on $\left(\frac{dx_1^*}{dk}\Big|_{k=0}, \ldots, \frac{dx_N^*}{dk}\Big|_{k=0}\right)$ as part (ii) of Proposition 5 predicts. This completes the proof. ■

Proof of Corollary 1

Proof. Suppose to the contrary that case (c) occurs for certain combination of (v_1, v_2, v_3). It follows immediately that $v_1 > v_2$. Moreover, all three players must be active under $k = 0$. Otherwise, $\frac{dx_1^*}{dk}\Big|_{k=0} < 0$ and $\frac{dx_2^*}{dk}\Big|_{k=0} \geq 0$ cannot hold from Proposition 3.

Without loss of generality, we normalize $v_1 = 1$. It follows from Lemma 1 that all three contestants remain active under $k = 0$ requires that

$$\frac{2}{v_i} < \sum_{j=1}^{3} \frac{1}{v_j}, \forall\, i \in \{1, 2, 3\} \Rightarrow \frac{1}{v_3} < \frac{1}{v_2} + 1. \tag{24}$$

Next, it follows from (22) that

$$\frac{dx_i^*}{dk}\Bigg|_{k=0} = -s(0) \times \left[1 - 2p_i^*(0)\right] \times \left[2\tilde{p} - p_i^*(0)\right],$$

where $\tilde{p} = 1 - \frac{2\sum_{j=1}^{3}\frac{1}{v_j^2}}{\left[\sum_{j=1}^{3}\frac{1}{v_j}\right]^2}$, $s(0) = \frac{2}{\sum_{j=1}^{3}\frac{1}{v_j}}$, and $p_i^*(0) = 1 - \frac{s(0)}{v_i}$. Therefore, for case (c) to occur, we must have that $2\tilde{p} \leq p_2^* \leq \frac{1}{2}$, which in turn implies that

$$3\left(\frac{1}{v_3}\right)^2 - \left(\frac{4}{v_2} + 2\right) \times \frac{1}{v_3} + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 \geq 0.$$

However, the above inequality cannot hold. To see this, note that

$$\begin{aligned} 3\left(\frac{1}{v_3}\right)^2 - \left(\frac{4}{v_2} + 2\right) \times \frac{1}{v_3} + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 &< \left(\frac{1}{v_2} + 1\right)\left(\frac{3}{v_2} + 3 - \frac{4}{v_2} - 2\right) + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 \\ &< 4\left(1 - \frac{1}{v_2}\right) < 0, \end{aligned}$$

where the first inequality follows from $\frac{1}{3}(\frac{2}{v_2}+1) < \frac{1}{v_2} \leq \frac{1}{v_3}$ and (24), and the second inequality follows from $v_2 < v_1 = 1$. This concludes the proof. ■

Proof of Proposition 6

Proof. Suppose to the contrary that $\left|\mathcal{M}(k)\right| \geq \left|\mathcal{M}(0)\right| + 1 \equiv m+1$. By Equation (4), we can obtain that

$$\begin{aligned}
\sum_{i=1}^{m+1} p_i^*(k) &= \sum_{i=1}^{m+1} \frac{\sqrt{(1-3k)^2\left[s(k)\right]^2 + 8k\left[s(k)\right]^2\left(1-k-\frac{s(k)}{v_i}\right)} - (1-3k)s(k)}{4ks(k)} \\
&= \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}\right)^2 - 8\frac{s(k)}{kv_i}}\right] \\
&> \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}\right)^2 - 8\frac{(1-k)v_{m+1}}{kv_i}}\right] \\
&= \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}-4\frac{v_{m+1}}{v_i}\right)^2 + \frac{16v_{m+1}(v_i - v_{m+1})}{v_i^2}}\right] \\
&\geq \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\left(1+\frac{1}{k}-4\frac{v_{m+1}}{v_i}\right)\right] \\
&= \sum_{i=1}^{m+1}\left(1-\frac{v_{m+1}}{v_i}\right) = \sum_{i=1}^{m}\left(1-\frac{v_{m+1}}{v_i}\right) \geq \sum_{i=1}^{m}\left(1-\frac{s(0)}{v_i}\right) = 1,
\end{aligned}$$

where the first inequality follows from contestant $m+1$'s participation constraint $s(k) < (1-k)v_{m+1}$ in Equation (4); the second inequality follows from $v_1 \geq \ldots \geq v_{m+1}$; the third inequality follows from Equation (4) and the fact that contestant $m+1$ is inactive under $k=0$; and the last equality follows immediately from the rearrangement of Equation (18). Clearly, the above inequality contradicts

$$\sum_{i=1}^{m+1} p_i^*(k) \leq \sum_{i=1}^{\left|\mathcal{M}(k)\right|} p_i^*(k) \leq \sum_{i=1}^{N} p_i^*(k) = 1.$$

This completes the proof. ■

Proof of Proposition 7

Proof. Part (i) of the proposition is straightforward. Clearly, we must have $\left|\mathcal{M}(k)\right| \geq 2$. Moreover, it follows from Proposition 6 that $m \equiv \left|\mathcal{M}(0)\right| = 2$ indicates that $\left|\mathcal{M}(k)\right| \leq 2$. Therefore, we must have $\left|\mathcal{M}(k)\right| = 2$; and the equilibrium effort profile for the two active contestants is fully characterized by part (i) of Proposition 3, from which we can see that $x_1^*(k) + x_2^*(k)$ is a constant.

Next, we prove part (ii) of the proposition. The proof for the case where $m = 2$ and $v_1 > v_2$ is straightforward; and it remains to prove the result for the case $m \geq 3$. Note that the set of active contestants for a sufficiently small $k > 0$ is same as that for $k = 0$, i.e., $\left|\mathcal{M}(k)\right| = \left|\mathcal{M}(0)\right| \equiv m$ when k is small enough. The first-order condition $\frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = 0$ can be written as

$$\left[s(k) - x_i^*(k)\right] \times \left[s(k) - 2x_i^*(k)\right] kv_i + \left[s(k)\right]^3 - s(k)\left[s(k) - x_i^*(k)\right] v_i = 0, \text{ for all } i \in \{1, 2, \ldots, m\},$$

where $s(k) \equiv \sum_{i=1}^N x_i^*(k) = \sum_{i=1}^m x_i^*(k)$. Differentiating the above equation with respect k yields the following:

$$\begin{aligned}
&\left(\frac{ds}{dk} - \frac{dx_i^*}{dk}\right)(s - 2x_i^*)\, kv_i + (s - x_i^*)\left(\frac{ds}{dk} - 2\frac{dx_i^*}{dk}\right) kv_i \\
&+ (s - x_i^*)(s - 2x_i^*)\, v_i + 3s^2\frac{ds}{dk} - \frac{ds}{dk}(s - x_i^*)\, v_i - s\left(\frac{ds}{dk} - \frac{dx_i^*}{dk}\right) v_i = 0, \text{ for all } i \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Evaluating the above equation at $k = 0$, we can obtain

$$\begin{aligned}
&\left[s(0) - x_i^*(0)\right] \times \left[s(0) - 2x_i^*(0)\right] + \frac{3}{v_i}\left[s(0)\right]^2 \left.\frac{ds}{dk}\right|_{k=0} \\
&- \left.\frac{ds}{dk}\right|_{k=0} \times \left[s(0) - x_i^*(0)\right] - s(0)\left(\left.\frac{ds}{dk}\right|_{k=0} - \left.\frac{dx_i^*}{dk}\right|_{k=0}\right) = 0, \text{ for all } i \in \{1, 2, \ldots, m\}. \quad (25)
\end{aligned}$$

Summing up all the above conditions in (25) yields

$$(m-3)\left[s(0)\right]^2 + 2\sum_{i=1}^m \left[x_i^*(0)\right]^2 + 3\left[s(0)\right]^2 \left.\frac{ds}{dk}\right|_{k=0} \sum_{i=1}^m \frac{1}{v_i} - 2(m-1)s(0)\left.\frac{ds}{dk}\right|_{k=0} = 0,$$

which is equivalent to

$$\left.\frac{ds}{dk}\right|_{k=0} = -\frac{(m-3)\left[s(0)\right]^2 + 2\sum_{i=1}^m \left[x_i^*(0)\right]^2}{3\left[s(0)\right]^2 \sum_{i=1}^m \frac{1}{v_i} - 2(m-1)s(0)}. \quad (26)$$

It is evident the numerator is strictly positive for $m \geq 3$. Moreover, we have that

$$3\left[s(0)\right]^2 \sum_{i=1}^{m} \frac{1}{v_i} - 2(m-1)s(0) = 3\left[\frac{m-1}{\sum_{i=1}^{m}\frac{1}{v_i}}\right]^2 \sum_{i=1}^{m} \frac{1}{v_i} - 2(m-1)\frac{m-1}{\sum_{i=1}^{m}\frac{1}{v_i}} = \frac{(m-1)^2}{\sum_{i=1}^{m}\frac{1}{v_i}} > 0,$$

where the first equality follows from (18). This in turn implies that $\frac{ds}{dk}\big|_{k=0} < 0$ and completes the proof. ■

Proof of Proposition 8

Proof. Part (i) of the proposition is trivial, and it remains to prove part (ii). Recall that $\left|\mathcal{M}(k)\right| = \left|\mathcal{M}(0)\right| \equiv m$ for a sufficiently small $k > 0$ from the proof of Proposition 7. Combining $p_i^*(k) = x_i^*(k)/s(k)$ and the first-order condition $\frac{\widehat{U}_i(x_i,\boldsymbol{x}_{-i})}{\partial x_i} = 0$, we have that

$$2k(p_i^*)^2 + p_i^*(1-3k) - 1 + k + \frac{s}{v_i} = 0, \text{ for all } i = 1, \ldots, m.$$

Differentiating the above equation with respect to k and rearranging yield

$$\frac{dp_i^*(k)}{dk} = \frac{-2(p_i^*)^2 + 3p_i^* - 1 - \frac{1}{v_i}\frac{ds}{dk}}{4p_i^*k + 1 - 3k}, \text{ for all } i = 1, \ldots, m,$$

which in turn implies that

$$\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = -2\left[p_i^*(0)\right]^2 + 3p_i^*(0) - 1 - \frac{1}{v_i} \times \left.\frac{ds}{dk}\right|_{k=0}, \text{ for all } i = 1, \ldots, m. \tag{27}$$

Combing (17) and (18) yields that

$$p_i^*(0) = \frac{x_i^*(0)}{s(0)} = 1 - \frac{s(0)}{v_i}, \text{ for all } i = 1, \ldots, m. \tag{28}$$

Plugging (28) into (27) yields that

$$\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = -2\left[s(0)\right]^2 \times \frac{1}{v_i^2} + \left[s(0) - \left.\frac{ds}{dk}\right|_{k=0}\right] \times \frac{1}{v_i}, \text{ for all } i = 1, \ldots, m. \tag{29}$$

Moreover, we have that

$$\left.\frac{ds(k)}{dk}\right|_{k=0} = -\frac{(m-3)\left[s(0)\right]^2 + 2\sum_{i=1}^{m}\left\{s(0) - \frac{\left[s(0)\right]^2}{v_i}\right\}^2}{3\left[s(0)\right]^2\sum_{i=1}^{m}\frac{1}{v_i} - 2(m-1)s(0)}$$
$$= -\frac{(m-3)\left[s(0)\right]^2 + 2\sum_{i=1}^{m}\left\{s(0) - \frac{\left[s(0)\right]^2}{v_i}\right\}^2}{3(m-1)s(0) - 2(m-1)s(0)} = s(0) - \frac{2\left[s(0)\right]^3 \times \sum_{i=1}^{m}\frac{1}{v_i^2}}{m-1}, \tag{30}$$

where the first equality follows from Equation (26) in the proof of Proposition 7 and Equation (17); the second and the third equalities follow from Equation (18). Combining Equations (18), (29) and (30), it can be verified that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ is equivalent to

$$v_i > \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}.$$

Moreover, it can be verified that

$$v_1 > \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}, \text{ and } v_m < \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}.$$

Therefore, there exists a cutoff τ_p such that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ for $i \leq \tau_p$ and $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} \leq 0$ otherwise. This completes the proof. ■

Proof of Theorem 2

Proof. Recall that $\boldsymbol{y} \equiv (y_1, \ldots, y_N)$ in the proof of Theorem 1. Then the expected utility of contestant i in expression (2) can be rewritten as

$$\pi_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i}) = \frac{y_i}{\sum_{j=1}^{N} y_j} v_i \times \left[(1+\eta) + \eta(\lambda - 1)\frac{\hat{y}_i}{\sum_{j\neq i} y_j + \hat{y}_i}\right]$$
$$- \phi_i(y_i) + \eta\mu\left(\phi_i(\hat{y}_i) - \phi_i(y_i)\right) - \eta\lambda\frac{\hat{y}_i}{\sum_{j\neq i} y_j + \hat{y}_i} v_i, \tag{31}$$

where $y_i := f_i(x_i)$ and $\hat{y}_i := f_i(\hat{x}_i)$. To prove the existence and uniqueness of PPNE of the original contest game, it is equivalent to show that there exists a unique PPNE of the modified contest game in which contestant $i \in \mathcal{N}$ chooses $y_i \geq 0$ simultaneously and his utility function is given by Equation (31).

Note that $\pi_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})$ is strictly concave in y_i for $y_i > \hat{y}_i$ and $y_i < \hat{y}_i$ respectively. Therefore, a sufficient and necessary condition for $\hat{y}_i > 0$ to be a personal equilibrium is

$$\left.\frac{\partial \pi_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})}{\partial y_i}\right|_{y_i \searrow \hat{y}_i} \leq 0, \text{ and } \left.\frac{\partial \pi_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})}{\partial y_i}\right|_{y_i \nearrow \hat{y}_i} \geq 0.$$

Carrying out the algebra, the above two inequalities are equivalent to

$$(1+\eta)\phi_i'(\hat{y}_i) \leq v_i \left[1 + \eta + \eta(\lambda - 1)\frac{\hat{y}_i}{\sum_{j\neq i}^N y_j + \hat{y}_i}\right] \times \frac{\sum_{j\neq i}^N y_j}{\left(\sum_{j\neq i}^N y_j + \hat{y}_i\right)^2} \leq (1+\eta\lambda)\phi_i'(\hat{y}_i).$$

For $s > 0$, let us define $\underline{g}_i(s)$ and $\overline{g}_i(s)$ as the following:

$$\underline{g}_i(s) = \begin{cases} 0 & \text{if } \frac{1+\eta}{1+\eta\lambda} v_i \leq \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right] = \frac{1+\eta\lambda}{v_i}\phi_i'(y_i) & \text{otherwise,} \end{cases} \tag{32}$$

and

$$\overline{g}_i(s) = \begin{cases} 0 & \text{if } v_i \leq \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right] = \frac{1+\eta}{v_i}\phi_i'(y_i) & \text{otherwise,} \end{cases} \tag{33}$$

Note that $\frac{s-y_i}{s^2} \times \left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right]$ is strictly decreasing in y_i given that $\eta(\lambda-1) \leq \frac{1}{2}$. Therefore, both $\underline{g}_i(s)$ and $\overline{g}_i(s)$ are well-defined; and it is straightforward to verify that $\underline{g}_i(s) \leq \overline{g}_i(s)$. Define $g_i^\dagger(s)$ as the following:

$$g_i^\dagger(s) = \begin{cases} \underline{g}_i(s) & \text{if } g_i(s) \leq \underline{g}_i(s), \\ g_i(s) & \text{if } \underline{g}_i(s) < g_i(s) < \overline{g}_i(s), \\ \overline{g}_i(s) & \text{if } g_i(s) \geq \overline{g}_i(s), \end{cases} \tag{34}$$

where $g_i(s)$ is defined in Equation (6) in the proof of Theorem 1. It can be verified that $\frac{\underline{g}_i(s)}{s}$ is strictly decreasing in s for $s < \frac{1}{\phi_i'(0)} \times \frac{1+\eta}{1+\eta\lambda} v_i$ and is equal to 0 otherwise. Similarly, $\frac{\overline{g}_i(s)}{s}$ is strictly decreasing in s for $s < \frac{1}{\phi_i'(0)} \times v_i$ and is equal to 0 otherwise. Recall that $\frac{g_i(s)}{s}$ is strictly decreasing in s for $s < \frac{1-k}{\phi_i'(0)} \times v_i$ and is equal to 0 otherwise. Therefore, $\frac{g_i^\dagger(s)}{s}$ is strictly decreasing in s for $s < \frac{v_i}{\phi_i'(0)}$ and is equal to 0 otherwise.

Note that $\widehat{\pi}_i(\boldsymbol{y})$ is strictly concave in y_i for all $i \in \mathcal{N}$ under Assumption 2. Therefore, the profile $\boldsymbol{y}^{**} \equiv (y_1^{**}, \ldots, y_N^{**})$ constitutes a PPNE of the modified contest game if and only if $s^{**} = \sum_{i=1}^N y_i^{**}$ satisfies

$$\sum_{i=1}^{N} \frac{g_i^\dagger(s^{**})}{s^{**}} = 1,$$

and then

$$y_i^{**} = g_i^\dagger\left(s^{**}\right), \text{ for all } i \in \mathcal{N}.$$

Therefore, it remains to show that there exists a unique positive solution to $\sum_{i=1}^N \frac{g_i^\dagger(s)}{s} = 1$, which follows instantly from the monotonicity of $\sum_{i=1}^N \frac{g_i^\dagger(s)}{s}$, and the facts that $\lim_{s\searrow 0} \sum_{i=1}^N \frac{g_i^\dagger(s)}{s} = N > 1$ and $\sum_{i=1}^N \frac{g_i^\dagger(s)}{s} = 0 < 1$ for $s \geq \frac{v_1}{\phi_i'(0)}$. This completes the proof. ■

Proof of Theorem 3

Proof. With slight abuse of notation, let us denote the CPNE under η by $\boldsymbol{x}^*(\eta) \equiv \left(x_1^*(\eta), \ldots, x_N^*(\eta)\right)$; and let $y_i^*(\eta) := f_i\left(x_i^*(\eta)\right)$ and $s^*(\eta) := \sum_{i=1}^N y_i^*(\eta)$ for all $i \in \mathcal{N}$. It suffices to verify that the unique CPNE is also a PPNE of the contest game when η is sufficiently small, holding fixed $\lambda > 1$. It can be verified that the set of active contestants under a sufficiently small η coincides with the set of active contestants under $\eta = 0$. Without loss of generality, we assume that the contestants are ordered with respect to $\frac{v_i}{\phi_i'(0)}$, that is,

$$\frac{v_1}{\phi_1'(0)} \geq \ldots \geq \frac{v_N}{\phi_N'(0)}.$$

Then there exists a cutoff $\widehat{\tau}$ such that $x_i^*(0) > 0$ for $i \leq \widehat{\tau}$ and $x_i^*(0) = 0$ otherwise.

1. [leftmargin=*]

i. For the active contestant $i \in \{1, \ldots, \widehat{\tau}\}$, it suffices to show that $\underline{g}_i(s) < g_i(s) < \overline{g}_i(s)$ from Equations (6) and (34) as $\eta \searrow 0$, which is equivalent to

$$\frac{s^*(\eta) - y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1 - \eta(\lambda-1) + 2\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right] > \frac{1}{1+\eta\lambda} \times \frac{s^*(\eta) - y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1 + \eta + \eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right],$$

and

$$\frac{s^*(\eta) - y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1 - \eta(\lambda-1) + 2\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right] < \frac{1}{1+\eta} \times \frac{s^*(\eta) - y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1 + \eta + \eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right],$$

from Equations (32) and (33). The first inequality is equivalent to

$$\frac{y_i^*(\eta)}{s^*(\eta)} > \frac{\eta\lambda}{1+2\eta\lambda},$$

which clearly holds as $\eta \searrow 0$ due to the fact that $\lim_{\eta\searrow 0}\frac{y_i^*(\eta)}{s^*(\eta)} = \frac{y_i^*(0)}{s^*(0)} > 0 = \lim_{\eta\searrow 0}\frac{\eta\lambda}{1+2\eta\lambda}$. Similarly, the second inequality can be simplified as

$$\frac{y_i^*(\eta)}{s^*(\eta)} < \frac{1+\eta}{1+2\eta},$$

which also holds as $\eta \searrow 0$ due to the fact that $\lim_{\eta\searrow 0}\frac{y_i^*(\eta)}{s^*(\eta)} = \frac{y_i^*(0)}{s^*(0)} < 1 = \lim_{\eta\searrow 0}\frac{1+\eta}{1+2\eta}$.

ii. For the inactive contestant $i \in \mathcal{N}\setminus\{1,\ldots,\widehat{\tau}\}$, note that $g_i\left(s^*(\eta)\right) = 0$ for a sufficiently small η. Together with Equation (6), $v_i \le \phi_i'(0) \times \frac{s^*(\eta)}{1-\eta(\lambda-1)}$ for $i \ge \widehat{\tau}+1$ as $\eta \searrow 0$. We consider the two following cases depending on $v_{\widehat{\tau}+1}$ relative to $\phi_{\widehat{\tau}+1}'^*(0)$.

(a) [leftmargin=*]

(a) $v_{\widehat{\tau}+1} < \phi_{\widehat{\tau}+1}'^*(0)$. Then $\frac{1+\eta}{1+\eta\lambda}v_i \le \phi_i'^*(\eta)$ must hold for a sufficiently small η. Otherwise, suppose to the contrary that $\frac{1+\eta}{1+\eta\lambda}v_i > \phi_i'^*(\eta)$ holds as $\eta \searrow 0$. Then we must have

$$v_i = \lim_{\eta\searrow 0}\frac{1+\eta}{1+\eta\lambda}v_i \ge \lim_{\eta\searrow 0}\left[\phi_i'^*(\eta)\right] = \phi_i'^*(0),$$

which is a contradiction to the postulated $v_i < \phi_i'^*(0)$. Therefore, we must have $\underline{g}_i\left(s^*(\eta)\right) = 0$ for a sufficiently small η from (32). It follows immediately from Equation (34) that $g_i^\dagger\left(s^*(\eta)\right) = g_i\left(s^*(\eta)\right) = 0$ for a sufficiently small η.

(b) $v_{\widehat{\tau}+1} = \phi_{\widehat{\tau}+1}'^*(0)$. We focus on the case in which $\frac{v_{\widehat{\tau}+1}}{\phi_{\widehat{\tau}+1}'(0)} > \frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$; the analysis for the case $\frac{v_{\widehat{\tau}+1}}{\phi_{\widehat{\tau}+1}'(0)} = \frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$ is similar. For ease of exposition, we let $v_{\widehat{\tau}+2} := 0$ if $\widehat{\tau}+2 > N$. Then there exists $\Delta > 0$ such that $\frac{v_{\widehat{\tau}+1}-\Delta}{\phi_{\widehat{\tau}+1}'(0)} > \frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$. Next, we consider the following vector of winning values:

$$\boldsymbol{v}_\Delta \equiv (v_1,\ldots,v_{\widehat{\tau}}, v_{\widehat{\tau}+1}-\Delta, v_{\widehat{\tau}+2},\ldots,v_N).$$

In words, all constants except contestant $\widehat{\tau}+1$ have the same winning values under $\boldsymbol{v} \equiv (v_1,\ldots,v_N)$ and $\boldsymbol{v}_\Delta$, whereas contestant $\widehat{\tau}+1$'s winning value under $\boldsymbol{v}_\Delta$ is strictly less than that under $\boldsymbol{v} \equiv (v_1,\ldots,v_N)$. It is straightforward to see that the unique CPNE under $\boldsymbol{v}_\Delta$ is the same as that under $\boldsymbol{v}$ as $\eta \searrow 0$ from Equation (6). Similarly, it can be verified that the unique PPNE under $\boldsymbol{v}_\Delta$ is the same as that under $\boldsymbol{v}$ as $\eta \searrow 0$ from Equations (6) and (32) to (34). Furthermore, the above analyses in part (a) imply instantly that the unique pure-strategy CPNE coincides with the unique pure-strategy PPNE under the profile of winning values $\boldsymbol{v}_\Delta$. Therefore, the unique pure-strategy CPNE is also the unique pure-strategy PPNE under the profile of winning values $\boldsymbol{v}$ as $\eta \searrow 0$. This completes the proof.

■

In this online appendix, we discuss the case of strong loss aversion, i.e.,$k \equiv \eta(\lambda - 1) > 1/3$.[1] We first show that CPNE may fail to exist when contestants are sufficiently k exceeds the cutoff 1/3. Next, we consider a simple contest design problem in which an effort-maximizing contest designer selects a contender to rival an incumbent player; the case sheds light on the the implications of loss aversion for contest design.

Existence and Uniqueness of CPNE

Let us introduce the notation $y_i := f_i(x_i)$, and define the inverse function of $f_i(\cdot)$ by $\phi_i(\cdot) := f_i^{-1}(\cdot)$. The function $\phi_i(\cdot)$ describes the amount of effort required for contestant i to generate an effective bid $y_i := f_i(x_i)$. We further assume the following.

Assumption A1 *$\phi_i(\cdot)$ is a trice differentiable function, with $\phi_i'(y_i) > 0$, $\phi_i''(y_i) \geq 0$, $\phi'''(y_i) \geq 0$, and $\phi_i(0) = 0$.*

Note that Assumption 1 implies immediately that $\phi_i'(y_i) > 0$, $\phi_i''(y_i) \geq 0$, and $\phi_i(0) = 0$. Compared to Assumption 1, the additional condition required by Assumption A1 is $\phi'''(y_i) \geq 0$, which is also assumed in Dato, Grunewald, and Müller (2018). Note that Assumption A1 is automatically satisfied if the impact function is linear.

Theorem A1 ***(Potential nonexistence of CPNE with large loss aversion)*** *Suppose that Assumption A1 is satisfied and $k \equiv \eta(\lambda - 1) \in [\frac{1}{3}, \frac{1}{2}]$. Then either (i) there exists a unique pure strategy CPNE of the contest game, or (ii) there exists no pure-strategy CPNE.*

Proof. The proof closely follows that of Theorem 1. We first show that $\rho_i'(s)$ in Equation (8) is positive, i.e.,

$$\rho_i'(s) = -\frac{\phi_i'(\rho_i s) + \rho_i s \times \phi_i''(\rho_i s)}{(1 - 3k + 4k\rho_i) v_i + s^2 \times \phi_i''(\rho_i s)} > 0,$$

Clearly, the numerator in the above expression is strictly positive due to the facts that $\phi_i' > 0$ and $\phi_i'' \geq 0$. For the denominator, we have

$$\begin{aligned}(1 - 3k + 4k\rho_i) v_i + s^2 \times \phi_i''(\rho_i s) &\geq (1 - 3k + 4k\rho_i) v_i + s \times \frac{\phi_i'(\rho_i s) - \phi_i'(0)}{\rho_i} \\ &> (1 - 3k + 4k\rho_i) v_i + \frac{1 - \rho_i}{\rho_i}(1 - k + 2k\rho_i) v_i - \frac{1}{\rho_i}(1 - k)v_i \\ &= 2k\rho_i v_i \geq 0,\end{aligned}$$

where the first inequality follows from $\phi_i''' \geq 0$ stated in Assumption A1; and the second inequality follows from (7) and $s < \frac{(1-k)v_i}{\phi_i'(0)}$.

[1]This note is not self-contained. It is the online appendix of the paper "Expectation-based Loss Aversion in Contests."

To complete the proof, it remains to show that $\chi(s) := \sum_{i=1}^{N} \rho_i(s) - 1 = 0$ has at most one positive solution for the case $\frac{1}{3} < k \leq \frac{1}{2}$. It can be verified that $\rho_i(s)$ is discontinuous at $s = (1-k)v_i/\phi_i'(0)$ for $\frac{1}{3} \leq k \leq \frac{1}{2}$. Moreover, $\rho_i(s)$ is continuous and strictly decreasing in s for $s < (1-k)v_i/\phi_i'(0)$, and is constant for $s \geq (1-k)v_i/\phi_i'(0)$. Therefore, $\chi(s)$ is strictly decreasing in s for $s \in (0, \frac{(1-k)v_1}{\phi_i'(0)}]$, but is discontinuous at $s = (1-k)v_i/\phi_i'(0)$ with $i \in \mathcal{N}$. This implies immediately that $\chi(s) = 0$ has at most one positive solution and concludes the proof. ■

Theorem A1 eliminates the possibility of multiple equilibria: whenever a CPNE exists, it must be unique. Interestingly, multiple CPNEs are possible in the framework of Dato, Grunewald, Müller, and Strack (2017). In particular, they show that an asymmetric equilibrium may exist when players are sufficiently loss averse, in which one player exerts no effort and the other player exerts positive effort. Such an equilibrium cannot arise in our framework due to the discontinuity of the contest success function at the origin.[2]

Theorem A1 also indicates a CPNE mail fail to exist when k exceeds 1/3. This is due to the fact that contestants' best response may display a discontinuity at a threshold of opponents' aggregate effort, and reaffirms the observation in Dato, Grunewald, Müller, and Strack (2017, Figure 1). Next, we provide two examples to briefly discuss equilibrium existence.

Example A1 ***(Existence of CPNE in contests with homogeneous players)*** *Suppose that Assumption 2 is satisfied and $k \in [0, \frac{1}{2}]$. Consider a contest that involves $N \geq 2$ homogeneous contestants with $v_1 = \ldots = v_N =: v > 0$ for all $i \in \mathcal{N}$.*

i. If $k \in [0, \frac{N}{3N-2}]$, then there exits a unique pure-strategy CPNE, in which all contestants exert an effort $x^* = \frac{N-1}{N^2}v - \frac{(N-1)(N-2)}{N^3}kv$.

ii. If $k \in (\frac{N}{3N-2}, \frac{1}{2}]$, then the contest game has no pure-strategy CPNE.

Part (ii) in the above example echoes Proposition 2 in Dato, Grunewald, Müller, and Strack (2017): When players are symmetric, there exists a threshold of the degree of loss aversion above which a CPNE fails to exist.

Next, we provide another example to illustrate the subtle impact of loss aversion on the existence of CPNE when contestant are heterogeneous.

Example A2 ***(Existence of CPNE in contests with asymmetric players)*** *Suppose that Assumption 2 is satisfied and $k \in [0, \frac{1}{2}]$. Consider a three-player contest with $(v_1, v_2, v_3) = (1, 0.9, 0.8)$. There exist two cutoffs $k_1 \approx 0.3650$ and $k_2 \approx 0.4098$ such that*

[2] To be more specific, once a contestant exerts zero effort, his opponent would sink infinitesimal small amount of effort to win the contest with probability one. This would both increase his material payoff and maximize the gain-loss utility by completely eliminating the underlying uncertainty of his realized payoff.

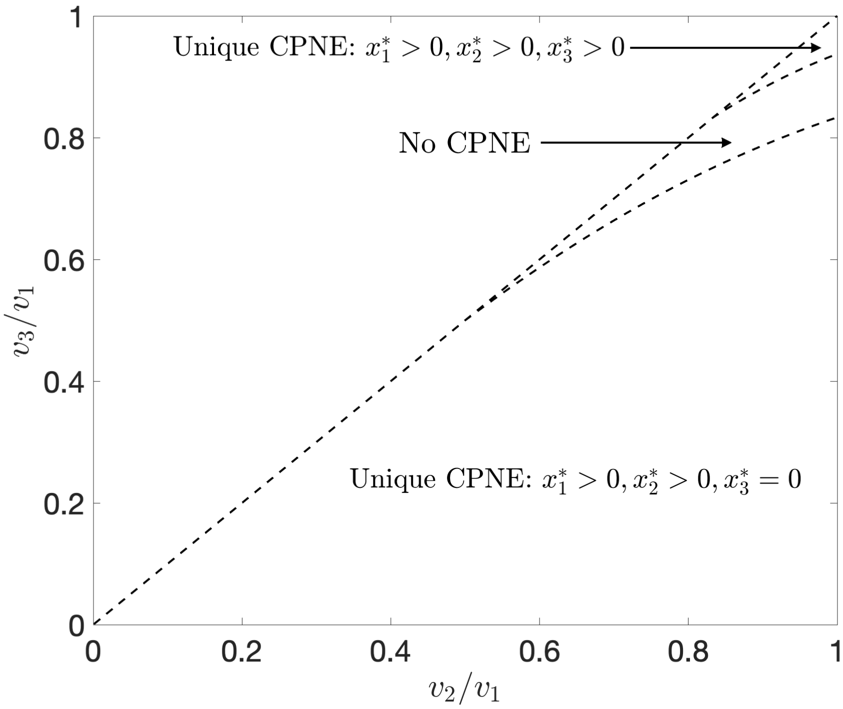

Figure 6: Existence of CPNE in Three-player Contests: $k = 0.4$.

i. For $k \in [0, k_1]$, there exists a unique pure-strategy CPNE, in which all three contestants exert a positive amount of effort.

ii. For $k \in (k_1, k_2)$, the contest game has no pure-strategy CPNE.

iii. For $k \in [k_2, \frac{1}{2}]$, there exists a unique pure-strategy CPNE, in which contestants 1 and 2 exert a positive amount of effort, whereas contestant 3 remains inactive.

In the same spirit, Figure 6 plots the combination of winning valuations (v_1, v_2, v_3) that lead to a unique CPNE or the nonexistence of CPNE in three-player contests with $k = 0.4$.

Contest Design: Contestant Selection

Thus far, we show that a large degree of loss aversion may cause equilibrium nonexistence. Next, we discuss the impact of loss aversion on contest design. To illuminate the implication of loss aversion on contest design most cleanly, we consider the following simple two-player contest design problem. It can be verified that a unique pure-strategy CPNE is guaranteed for all $k \in [0, \frac{1}{2}]$.

A contest designer is running a two-player contest and aims to maximize total effort. There exists an incumbent player whose valuation of winning the prize is normalized to one. The designer can select an opponent, denoted by $\hat{v}$, from a pool of talents/valuations $\mathcal{V} = [0, \infty]$. Denote the opponent's type in the optimal contest by $\hat{v}^*$. The following result can be established:

Proposition A1 ***(Optimal ability selection)*** *Suppose that Assumption 2 is satisfied and $k \in [0, \frac{1}{2}]$. Fix an arbitrary $\hat{v} \in \mathcal{V}$, there always exists a unique CPNE of the two-player contest game. Moreover, $\hat{v}^* = \infty$ if $k \in [0, \frac{1}{3}]$; and $1 < \hat{v}^* < \infty$ if $k \in (\frac{1}{3}, \frac{1}{2}]$.*

Proof. It is straightforward to verify that there exists a unique CPNE for all $\hat{v} > 0$ from Theorems 1 and A1, and the equilibrium effort profile is given by Proposition 2. Denote the total effort of inviting an contestant with winning valuation $\hat{v}$ by $TE(\hat{v})$. It follows from Proposition 2 that

$$TE(\hat{v}) = \frac{1}{1+\theta(\hat{v})} - \frac{1-\theta(\hat{v})}{\big[1+\theta(\hat{v})\big]^2}k,$$

where

$$\theta(\hat{v}) := \frac{1}{2}\left[\left(\frac{1}{\hat{v}} - 1\right) \times \frac{1+k}{1-k} + \sqrt{\left(\frac{1}{\hat{v}} - 1\right)^2 \times \left(\frac{1+k}{1-k}\right)^2 + \frac{4}{\hat{v}}}\right].$$

The first-order condition with respect to $\hat{v}$ yields

$$\frac{dTE}{d\hat{v}} = \frac{d\theta}{d\hat{v}} \times \left[-\frac{1}{(1+\theta)^2} + \frac{3-\theta}{(1+\theta)^3}k\right]$$

Carrying out the algebra, it is straightforward to verify that

$$\frac{d\theta}{d\hat{v}} = -\frac{1}{2} \times \frac{1}{\hat{v}^2} \times \left[\frac{1+k}{1-k} + \frac{\left(\frac{1}{\hat{v}} - 1\right) \times \left(\frac{1+k}{1-k}\right)^2 + 2}{\sqrt{\left(\frac{1}{\hat{v}} - 1\right)^2 \times \left(\frac{1+k}{1-k}\right)^2 + \frac{4}{\hat{v}}}}\right] < 0, \ \forall\, \hat{v} > 0.$$

Suppose that $k \le \frac{1}{3}$. Then we have that

$$\frac{dTE}{d\hat{v}} = \frac{d\theta}{d\hat{v}} \times \left[-\frac{1}{(1+\theta)^2} + \frac{3-\theta}{(1+\theta)^3}k\right] \ge -1 \times \frac{d\theta}{d\hat{v}} \times \frac{2}{3}\theta > 0, \ \forall\, \hat{v} > 0,$$

which indicates that $\hat{v}^* = \infty$.

Suppose that $k > \frac{1}{3}$. It can be verified that $\frac{dTE}{d\hat{v}} = 0$ is equivalent to

$$\theta(\hat{v}) = \frac{3k-1}{k+1}.$$

Recall that $\frac{d\theta}{d\hat{v}} < 0$. Moreover, $0 < \frac{3k-1}{k+1} < \frac{1}{3}$ for all $k \in (\frac{1}{3}, \frac{1}{2}]$, $\lim_{\hat{v}\searrow 1} \theta = 2$, and $\lim_{\hat{v}\nearrow\infty} \theta = 0$.

Therefore, there exists a unique solution to the above equation and thus $\hat{v}^* \in (1, \infty)$. This completes the proof. ■

By Proposition A1, an effort-maximizing contest designer will select an opponent that is moderately stronger than the incumbent to stimulate the incumbent when contestants are sufficiently loss averse. This result runs in stark contrast with the optimal ability selection problem with standard preferences. To see this more clearly, suppose that $k = 0$. In equilibrium, the incumbent exerts effort $\hat{v}/(1 + \hat{v})^2$ and the opponent exerts effort $\hat{v}^2/(1 + \hat{v})^2$. Simple algebra shows that total effort amounts to $\hat{v}/(1 + \hat{v})$, which is strictly increasing in $\hat{v}$. Therefore, the designer would select the strongest player from the pool of talent.